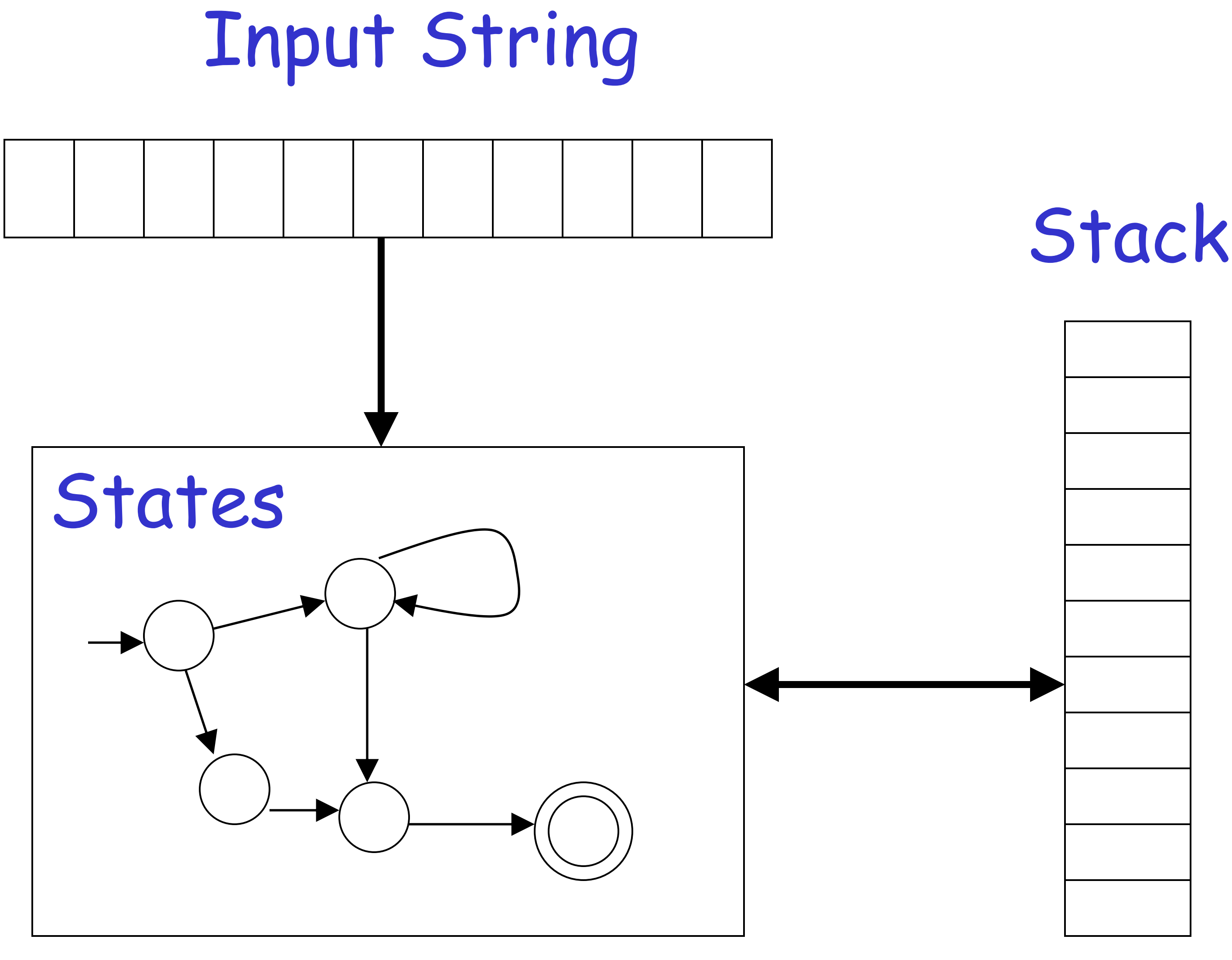
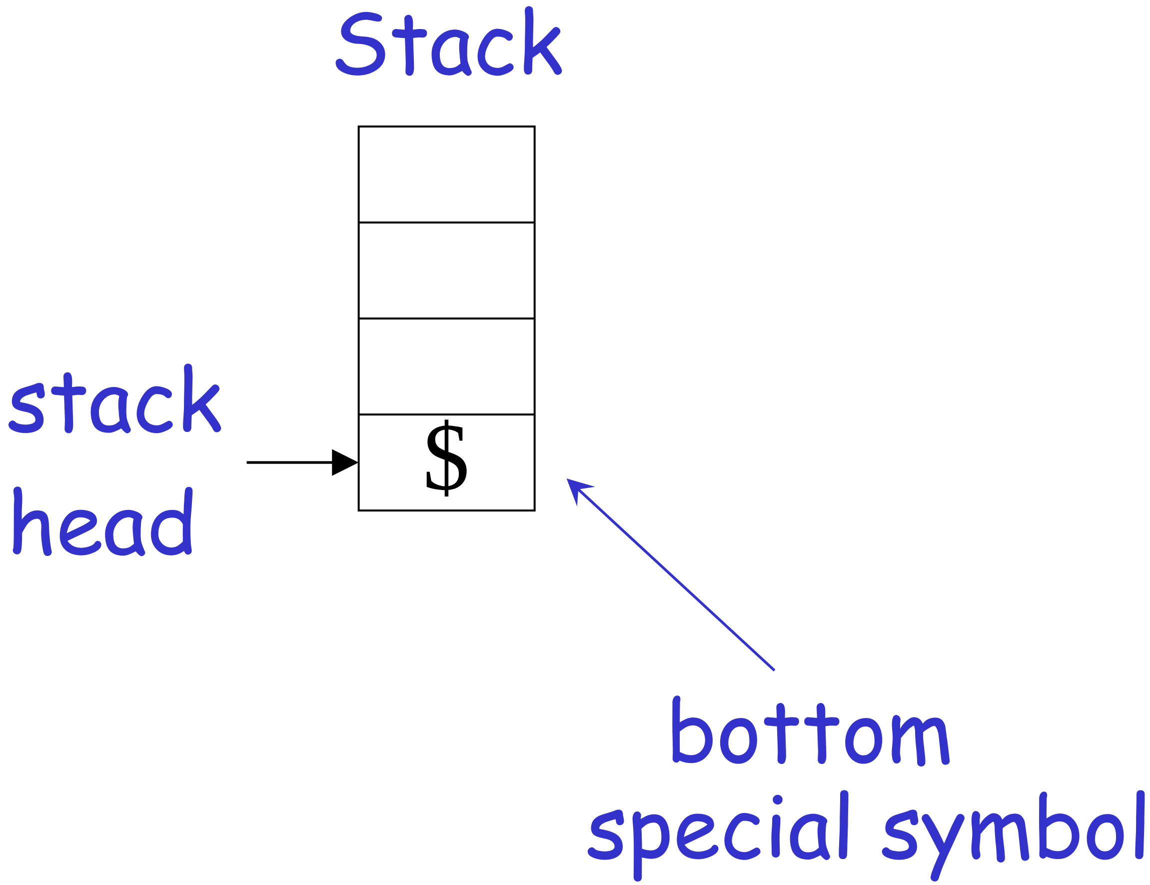
Pushdown Automata
PDAs

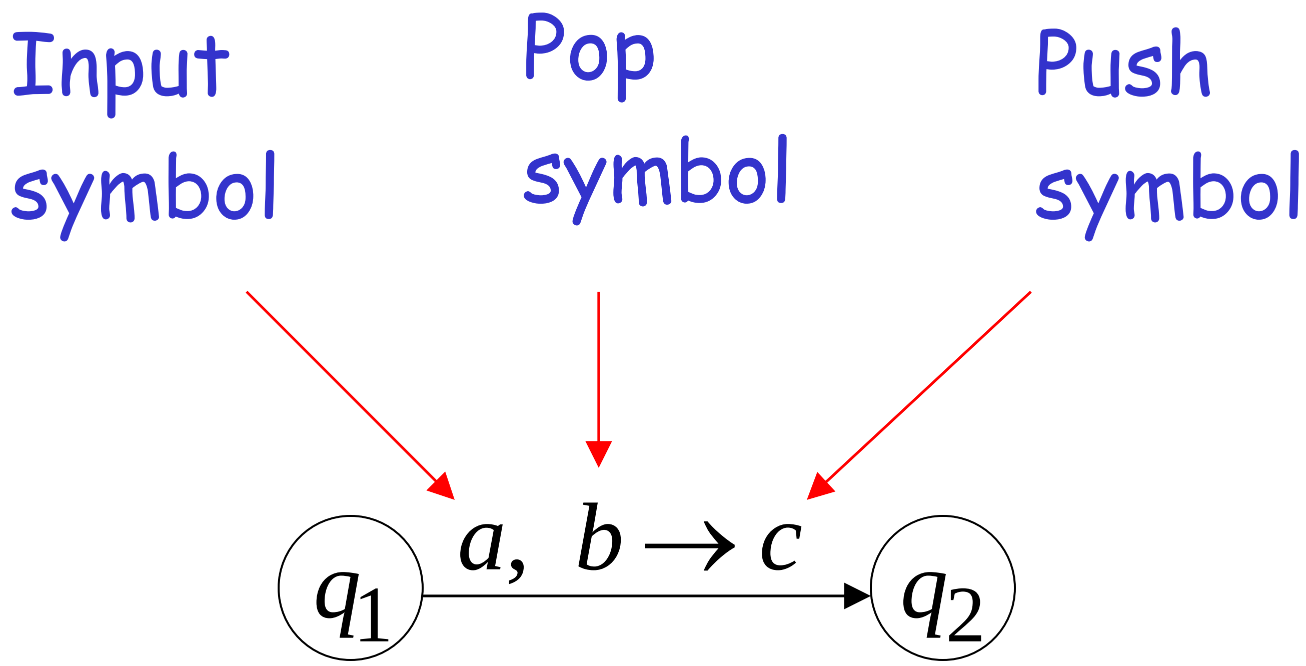
Pushdown Automaton -- PDA

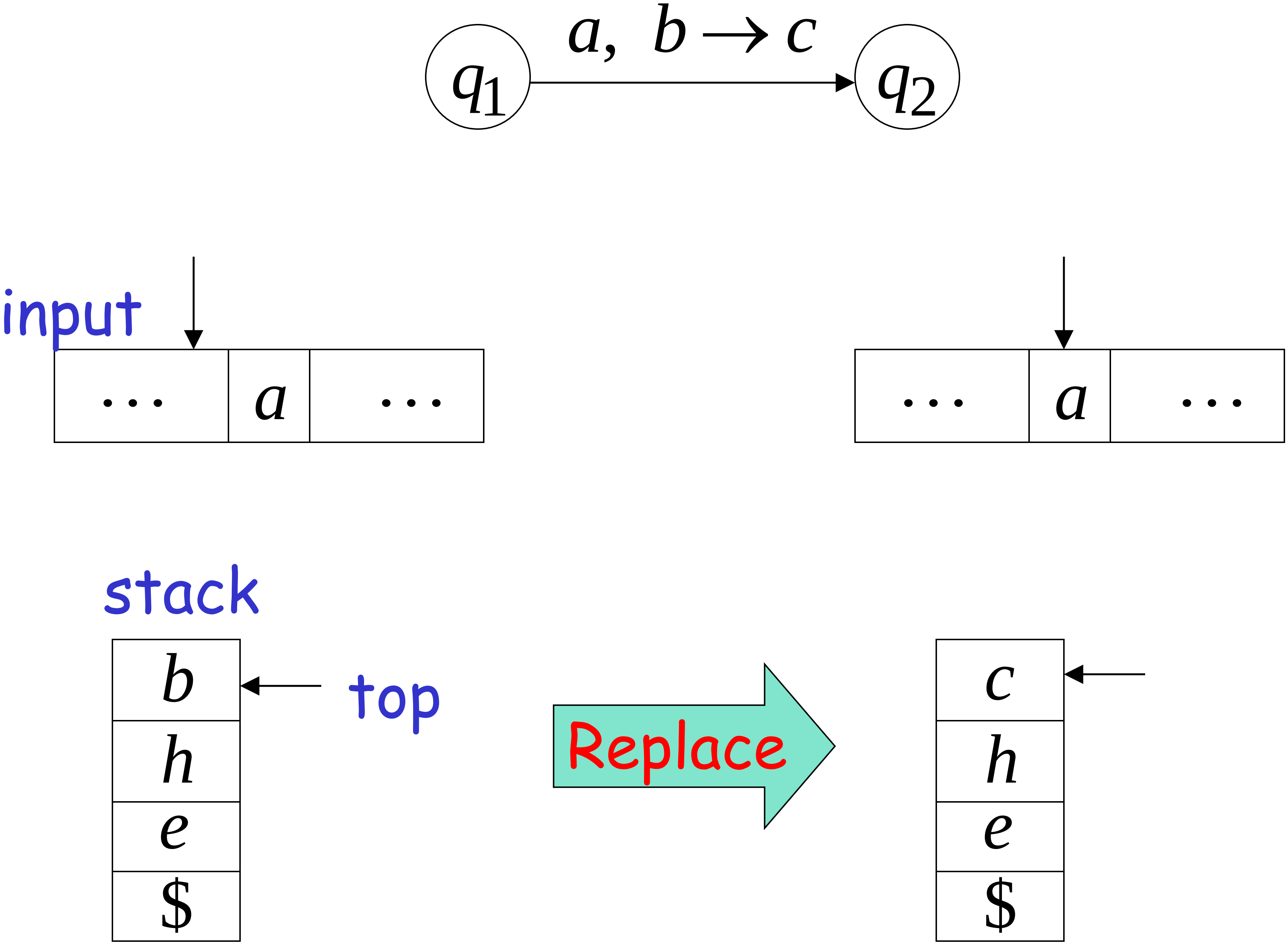


Initial Stack Symbol

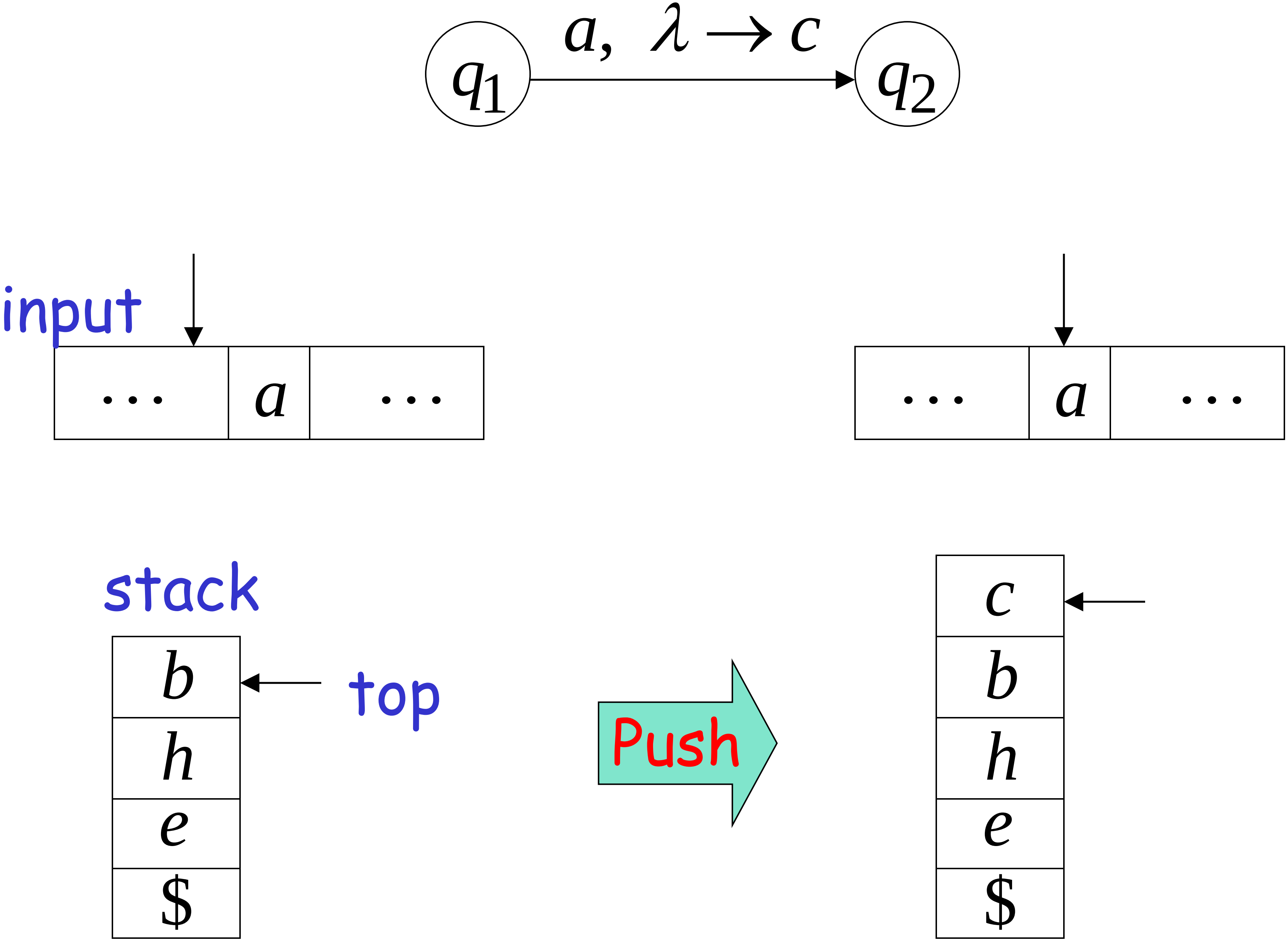


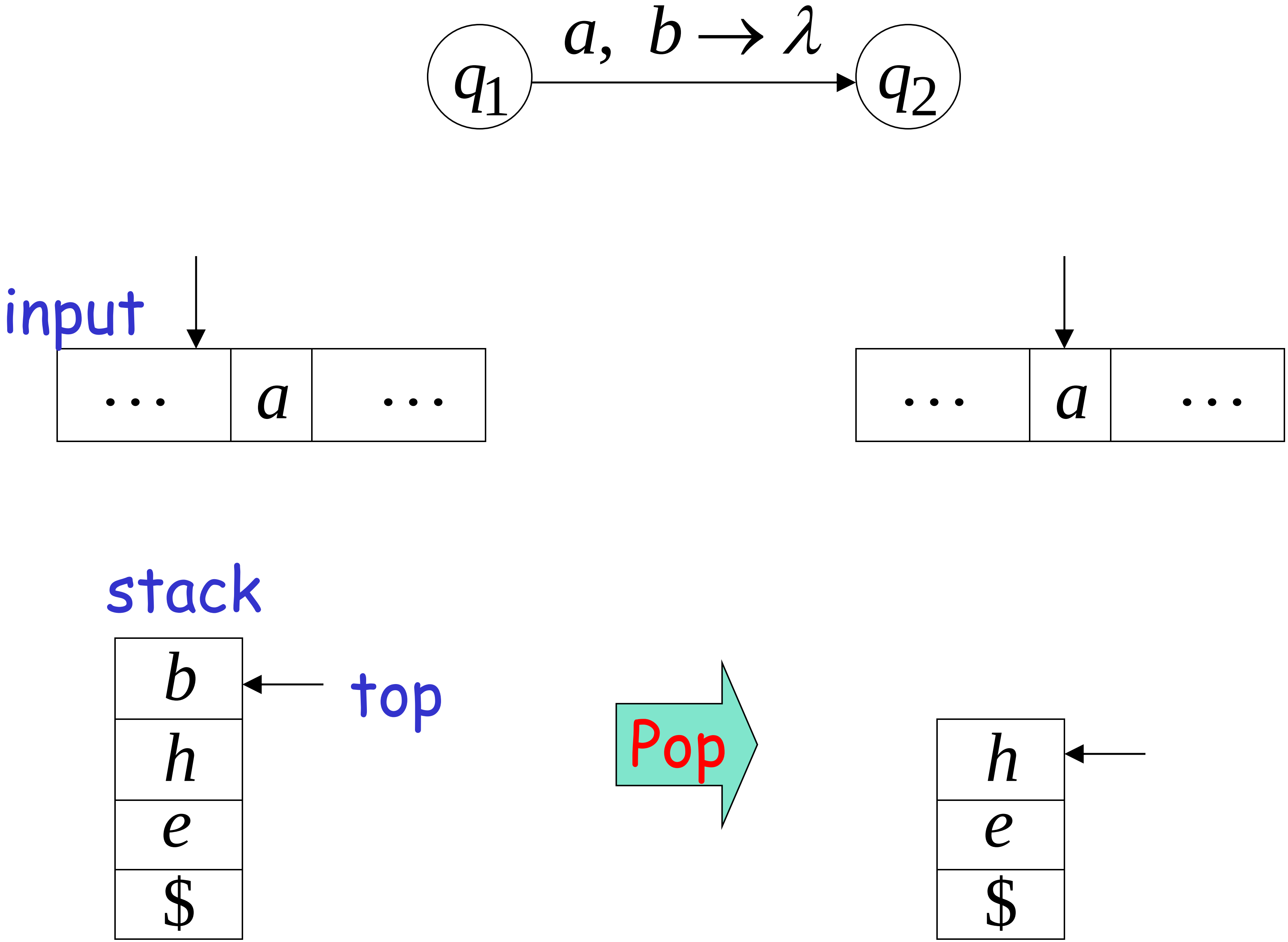
The States

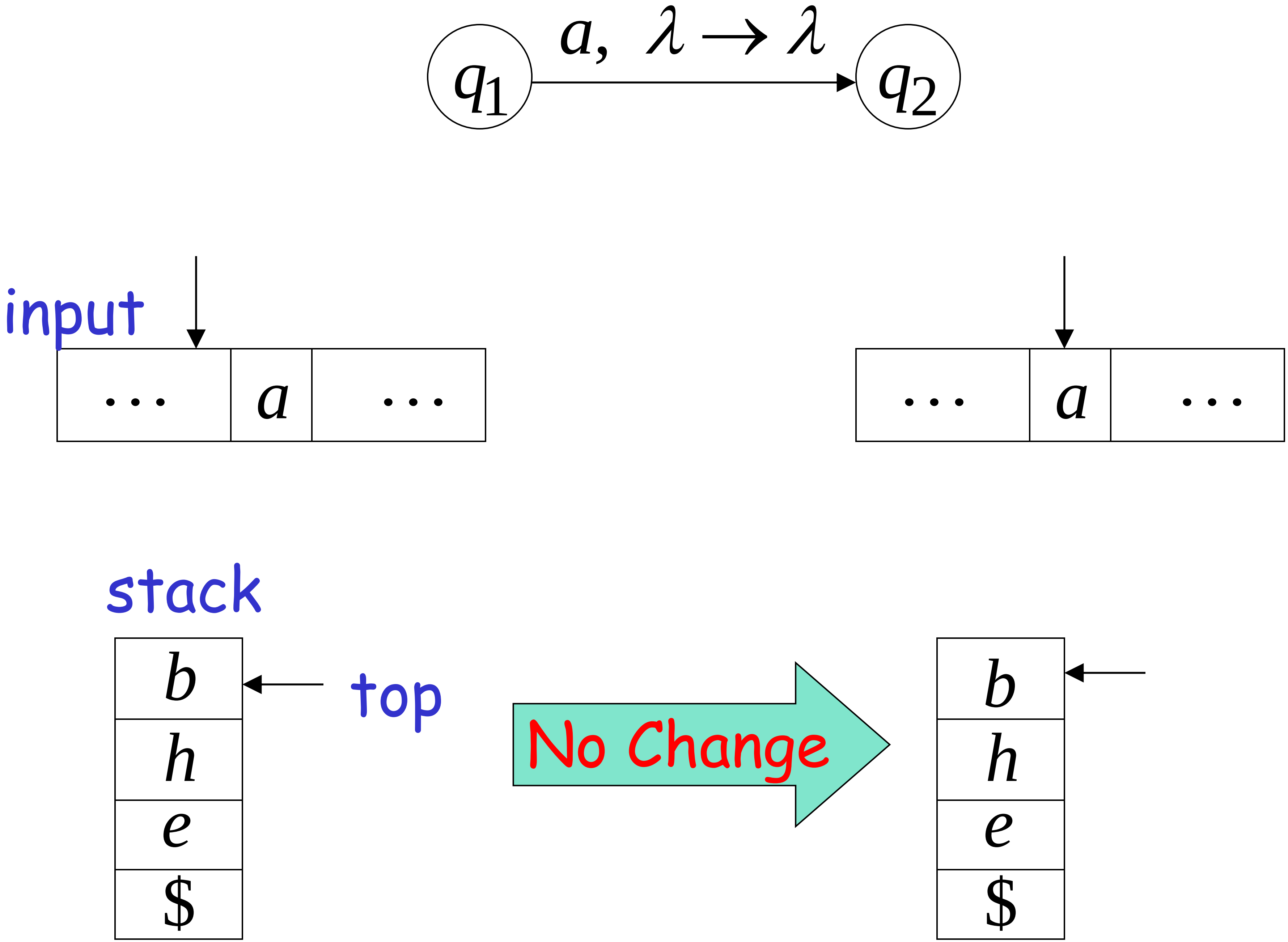




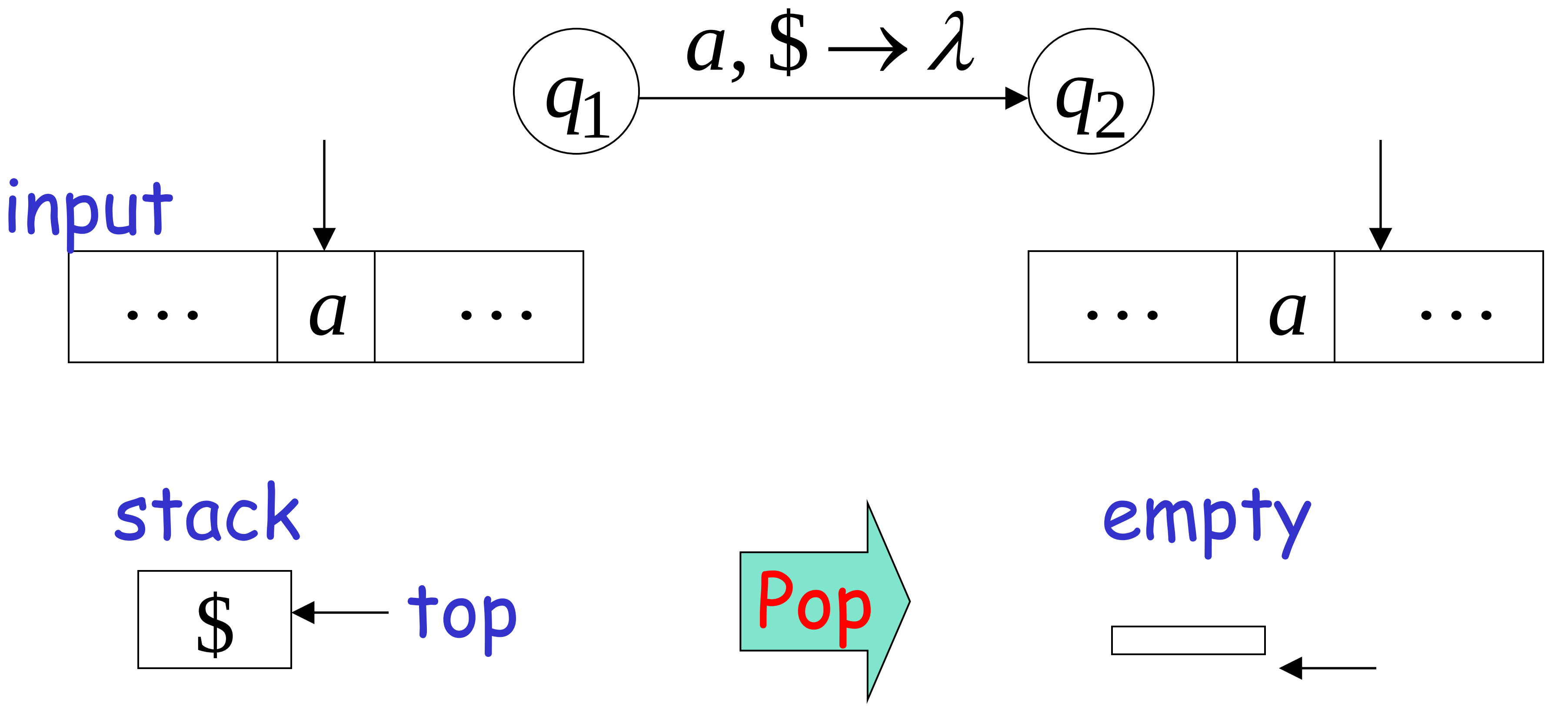
a, b → c
q1
q2
input
... a ...
... a ...
stack
b
h
e
$
top
Replace
c
h
e
$


q_1
$a, \lambda \rightarrow c$
q_2
input
$\cdots$ a $\cdots$
$\cdots$ a $\cdots$
stack
b
h
e
$\$$
top
Push
c
b
h
e
$\$$

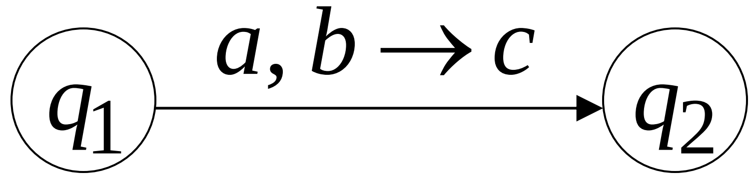

q_1
$a, \ b \to \lambda$
q_2
input
$\cdots$ a $\cdots$
$\cdots$ a $\cdots$
stack
top
b
h
e
$\$$
Pop
h
e
$\$$

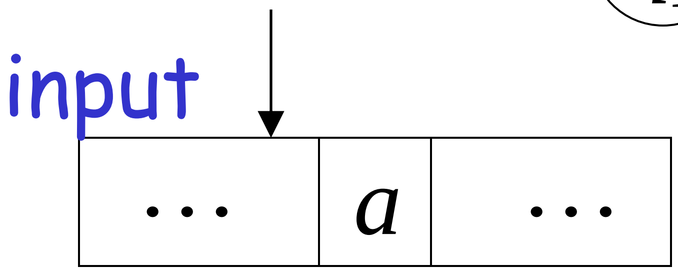

q_1
$a, \lambda \to \lambda$
q_2
input
$\cdots$ a $\cdots$
$\cdots$ a $\cdots$
stack
top
b h e $\$$
No Change
b h e $\$$

A Possible Transition

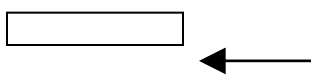


A Bad Transition





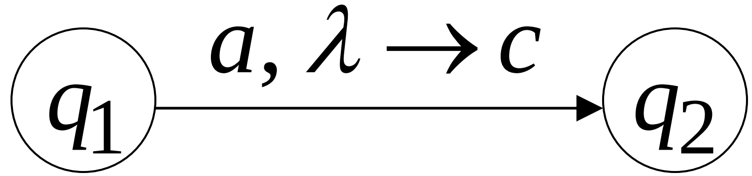
Empty stack

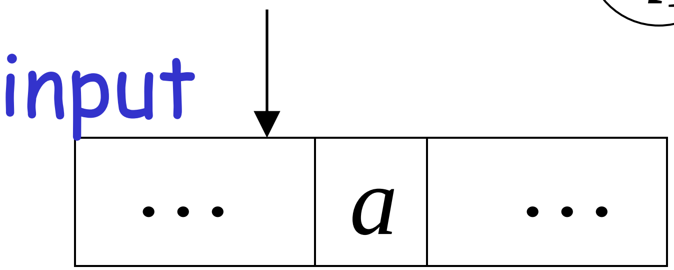


HALT

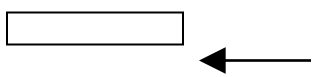
The automaton **Halts** in state q_1 and **Rejects** the input string

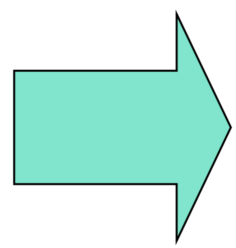
A Bad Transition





Empty stack



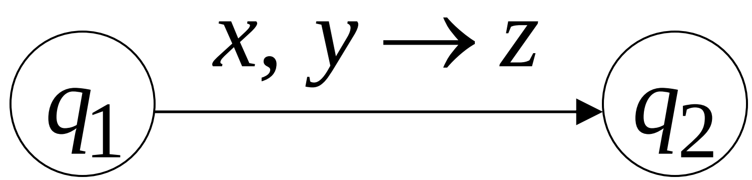


HALT

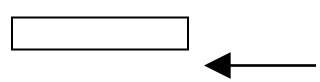
The automaton **Halts** in state q_1

and **Rejects** the input string

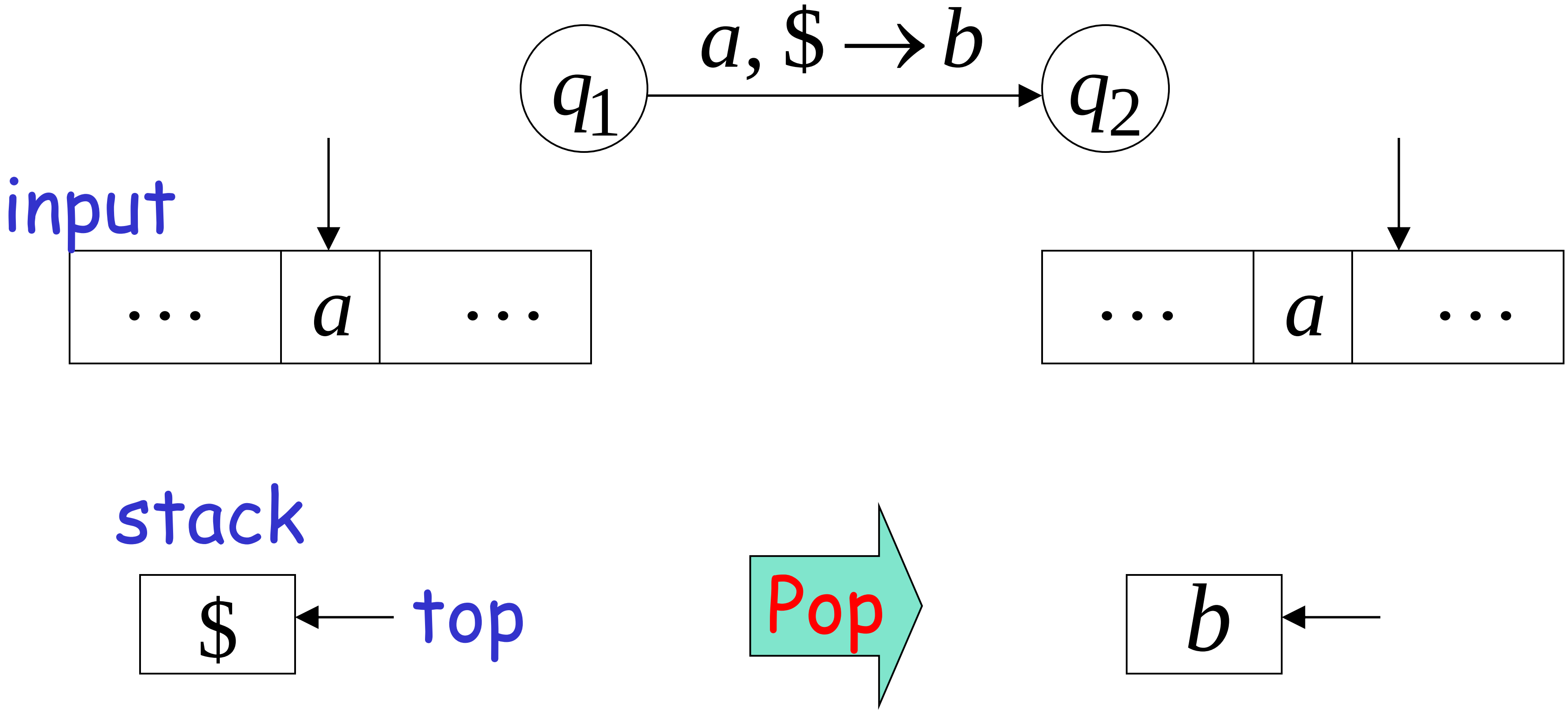
No transition is allowed to be followed
When the stack is empty



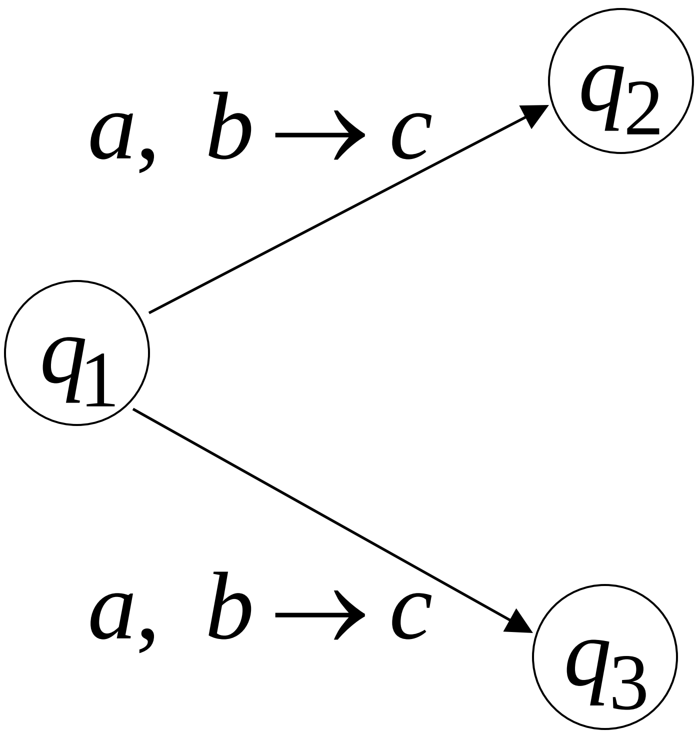
Empty stack

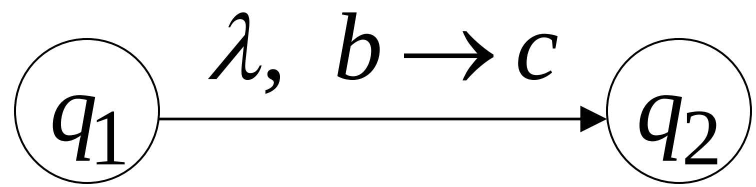


A Good Transition



Non-Determinism



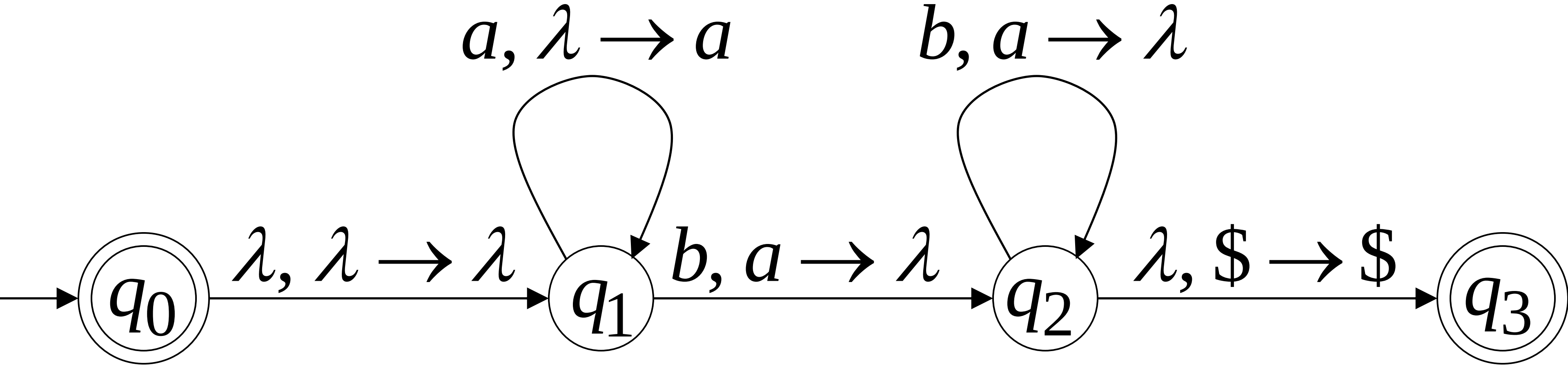


λ – transition

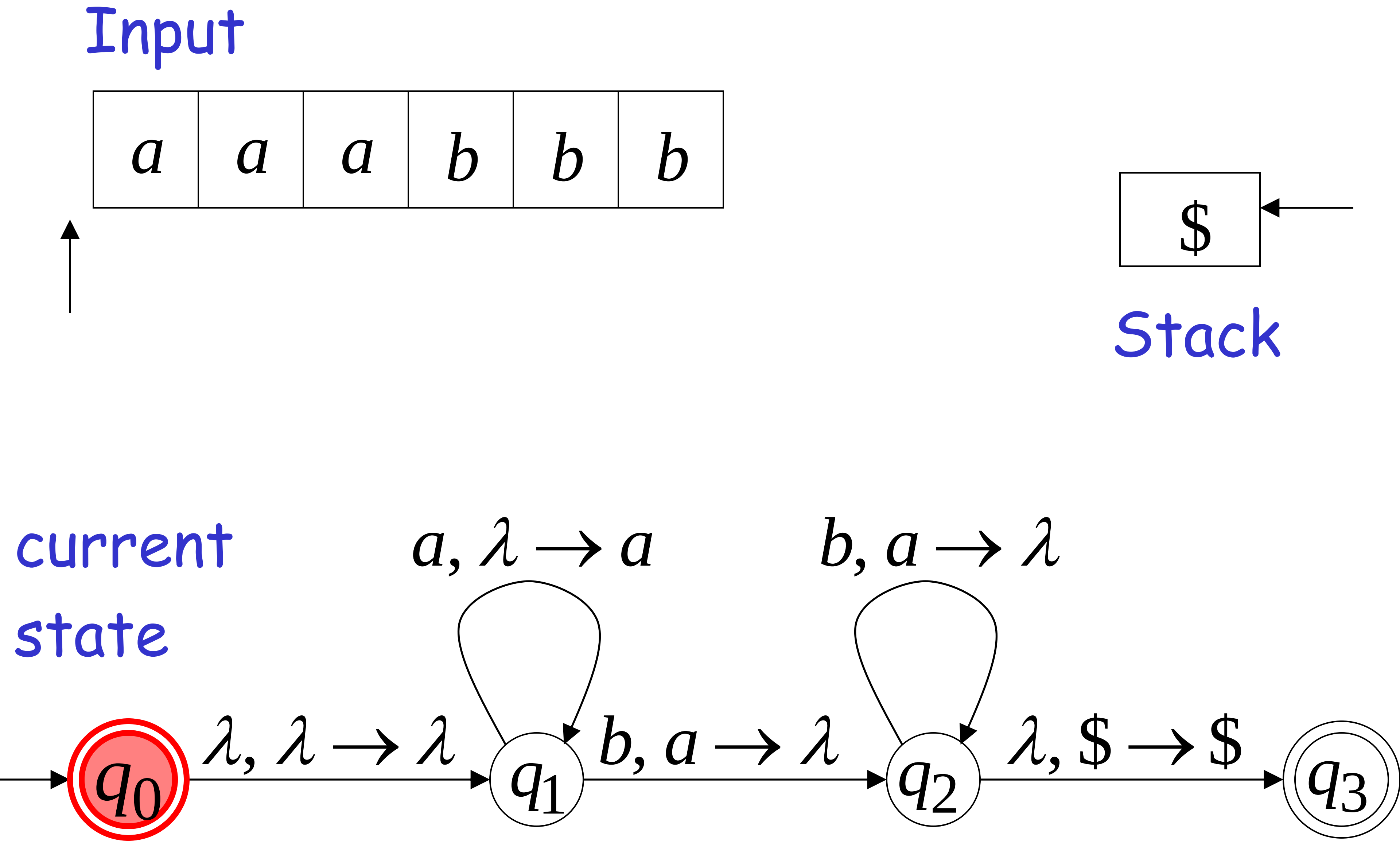
These are allowed transitions in a Non-deterministic PDA (NPDA)

NPDA: Non-Deterministic PDA

Example:



Execution Example: Time 0



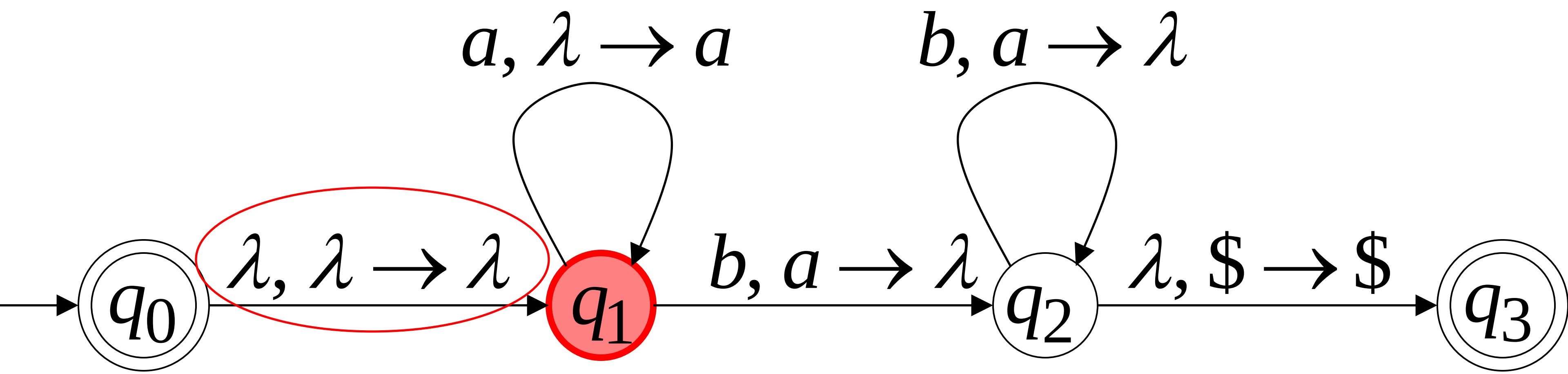
Time 1

Input

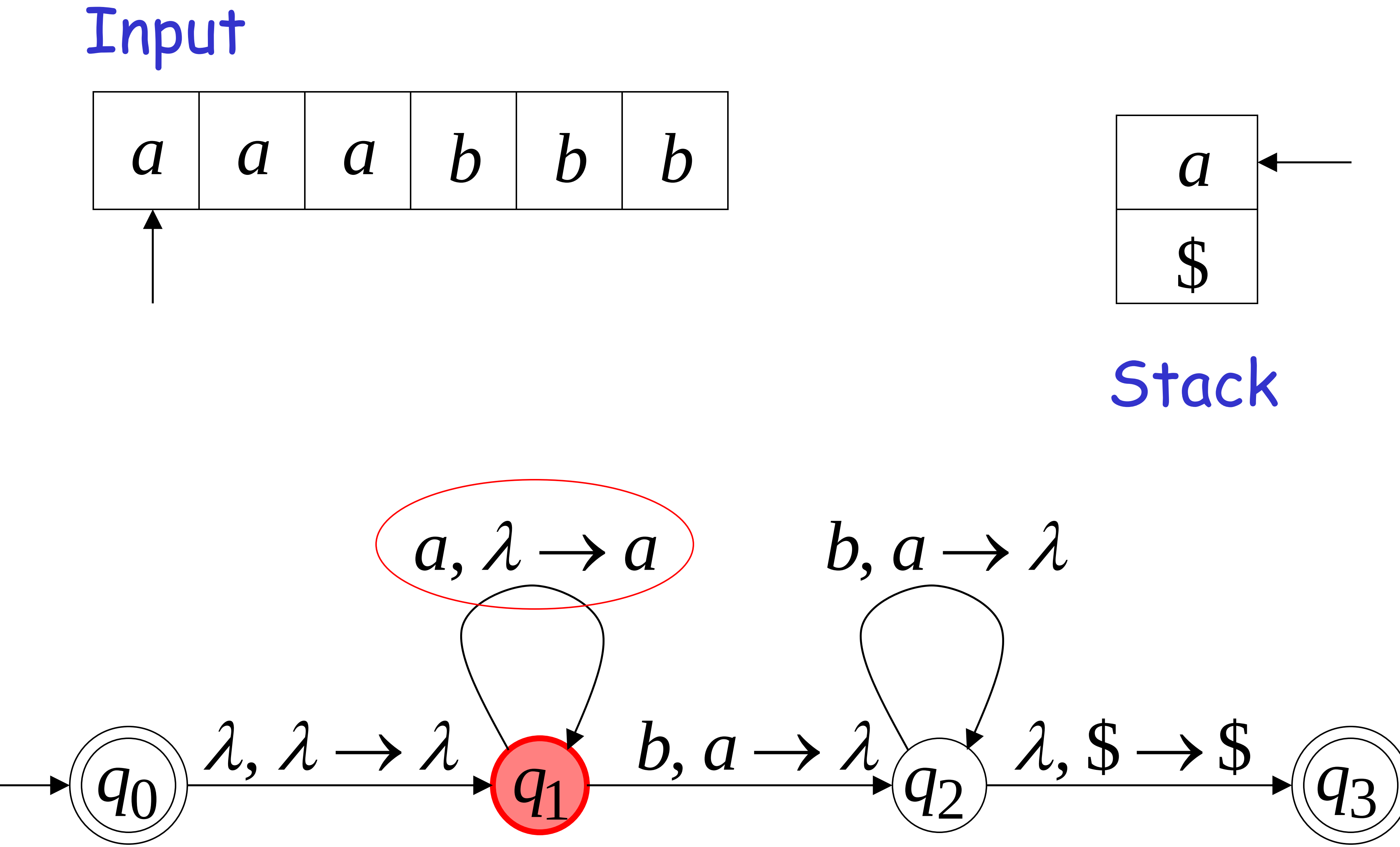
a	a	a	b	b	b

$\$$

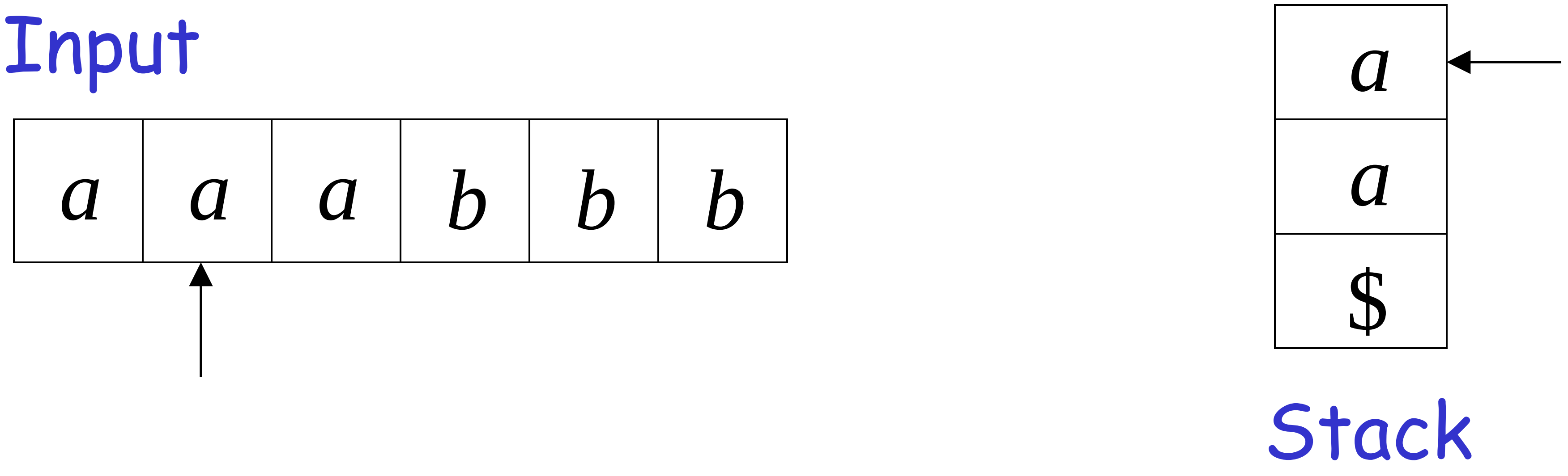
Stack

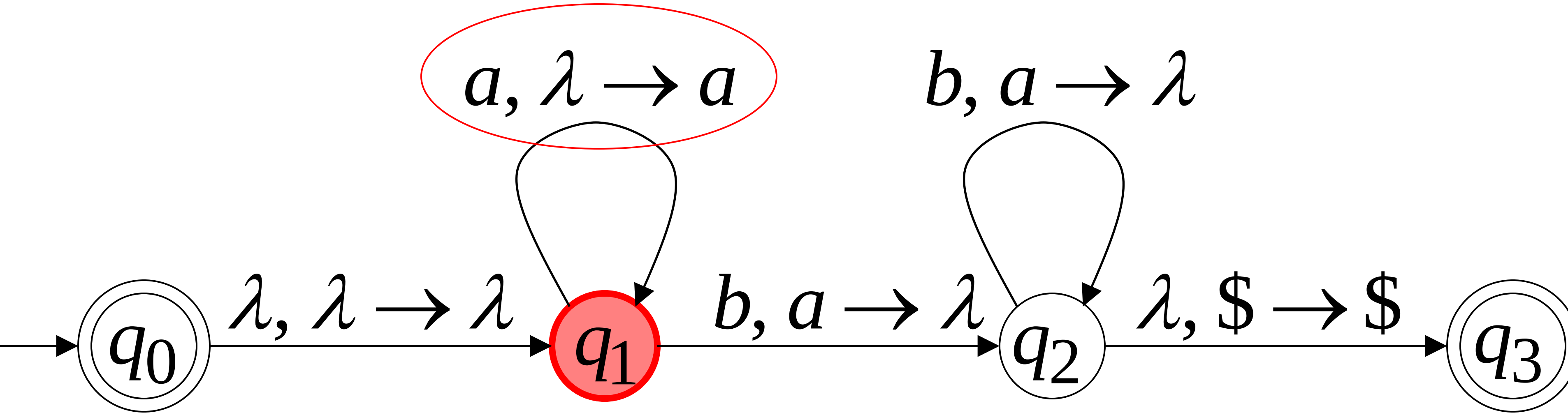


Time 2

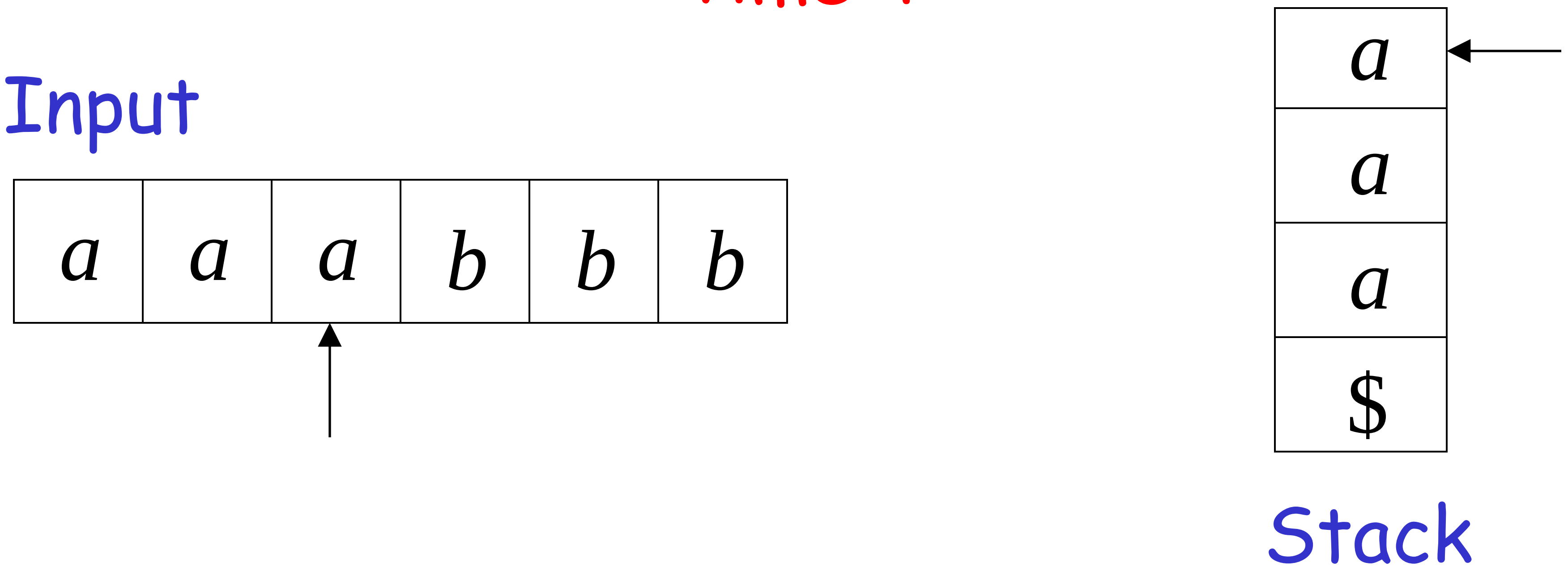

Input
a a a b b b
a
$
Stack
$a, \lambda \rightarrow a$
$b, a \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$b, a \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2
q_3

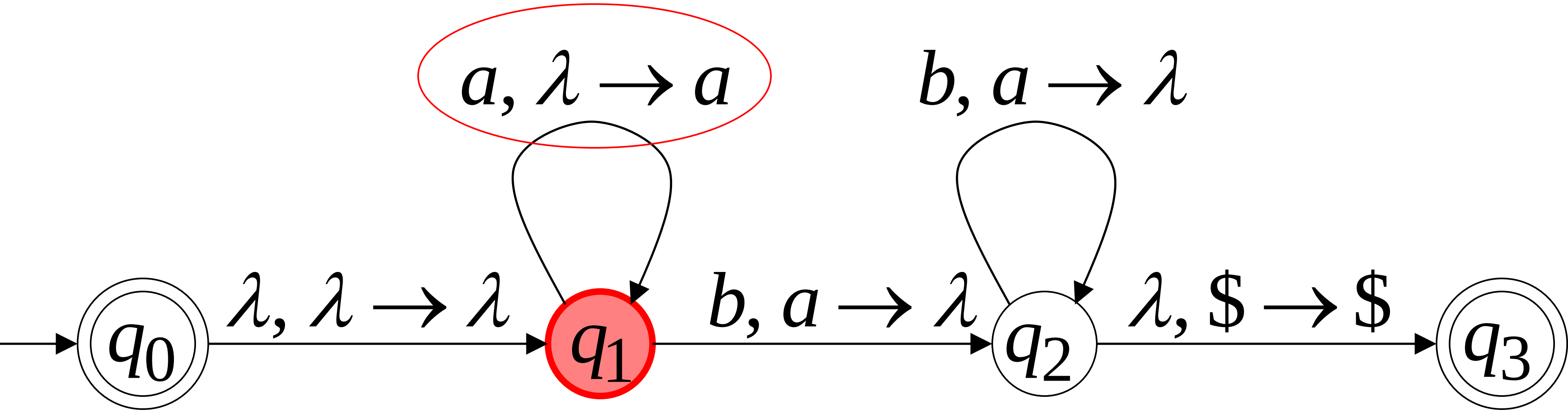
Time 3


Input
a a a b b b
a
a
$
Stack

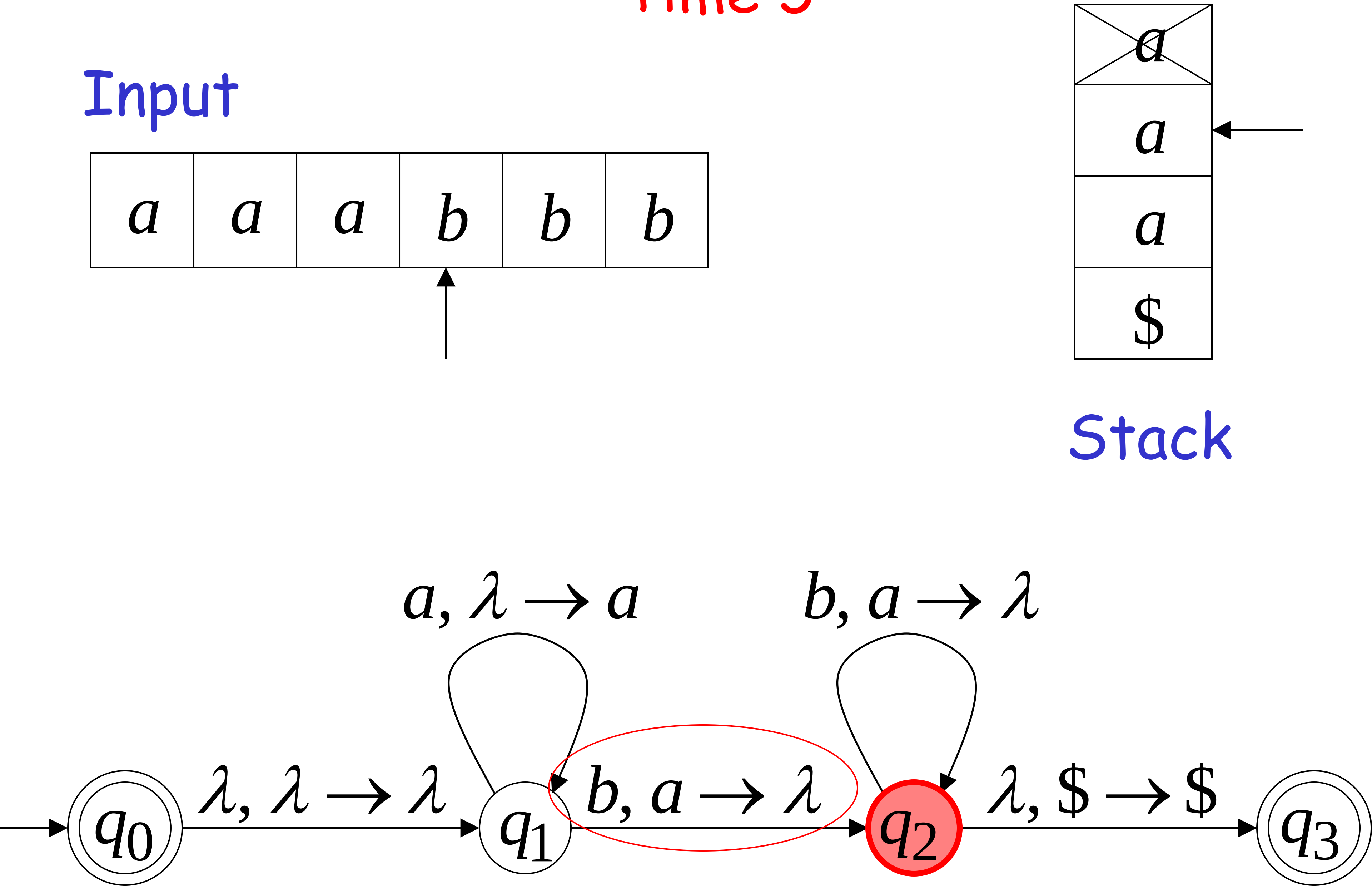

$a, \lambda \rightarrow a$
$b, a \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$b, a \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2
q_3

Time 4

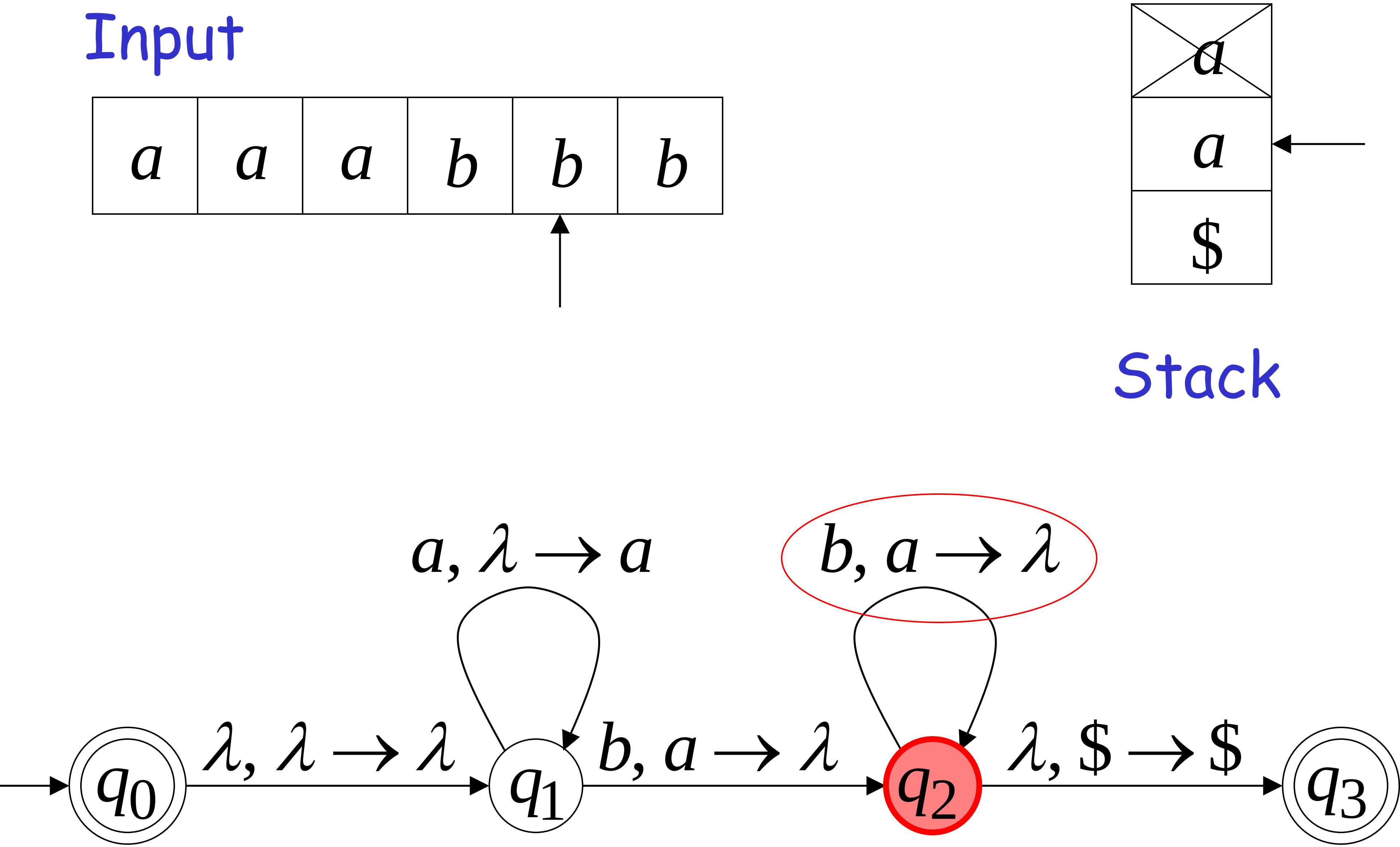

Input
a a a b b b
a
a
a
$
Stack


$a, \lambda \rightarrow a$
$b, a \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$b, a \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2
q_3

Time 5

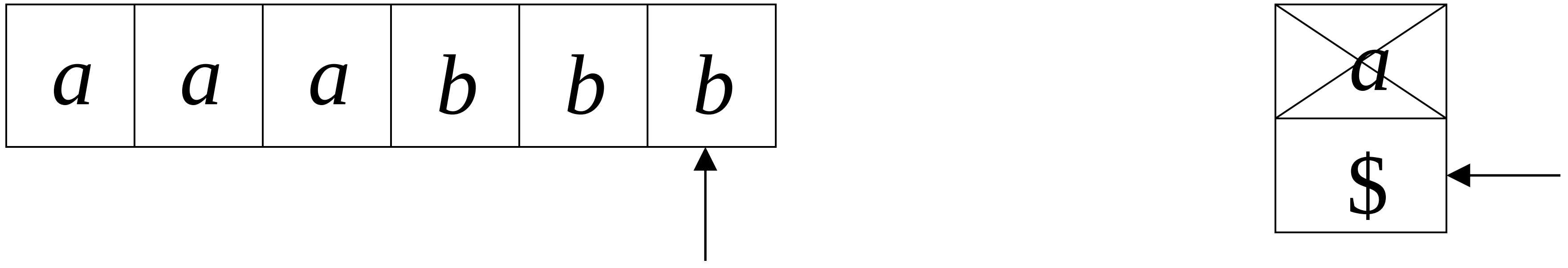

Input
a a a b b b
a
a
a
$
Stack
$a, \lambda \to a$
$b, a \to \lambda$
$\lambda, \lambda \to \lambda$
$b, a \to \lambda$
$\lambda, \$ \to \$$
q_0
q_1
q_2
q_3

Time 6

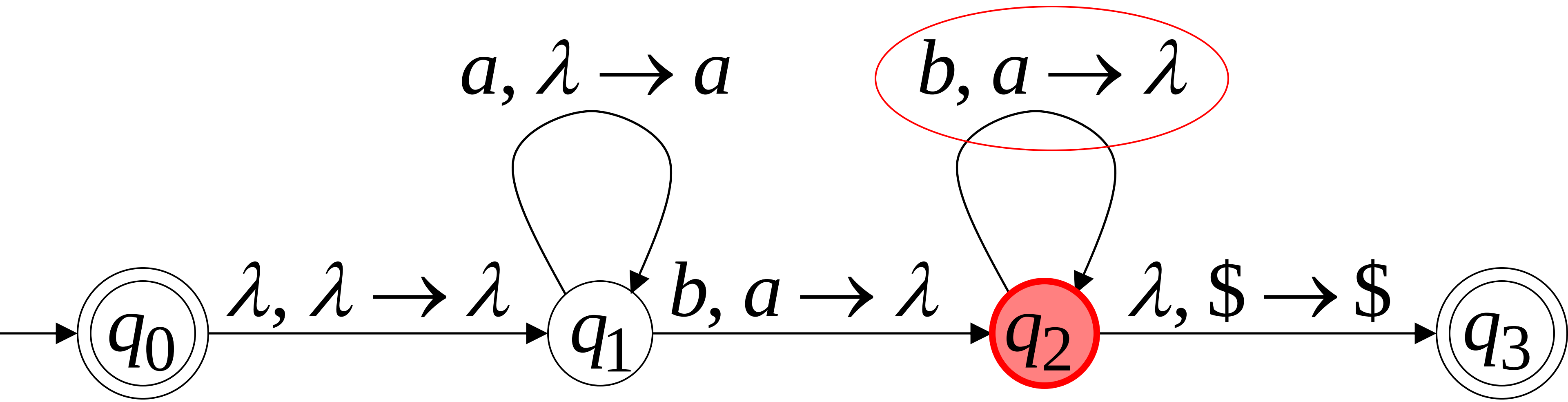

Input
a a a b b b
$\cancel{a}$
a
$\$$
Stack
$a, \lambda \rightarrow a$
$b, a \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$b, a \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2
q_3

Time 7

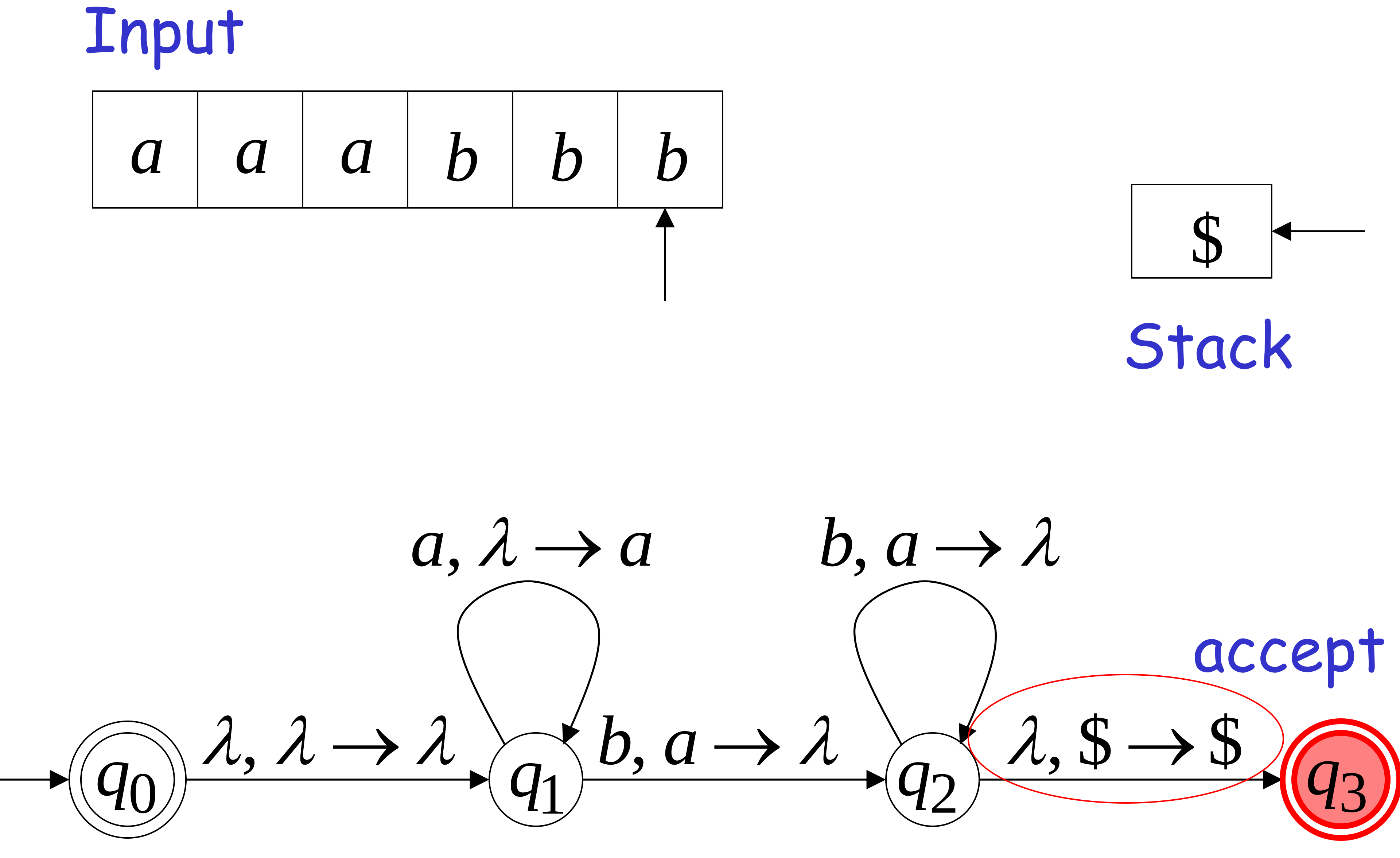
Input



Stack



Time 8



A string is accepted if there is
a computation such that:

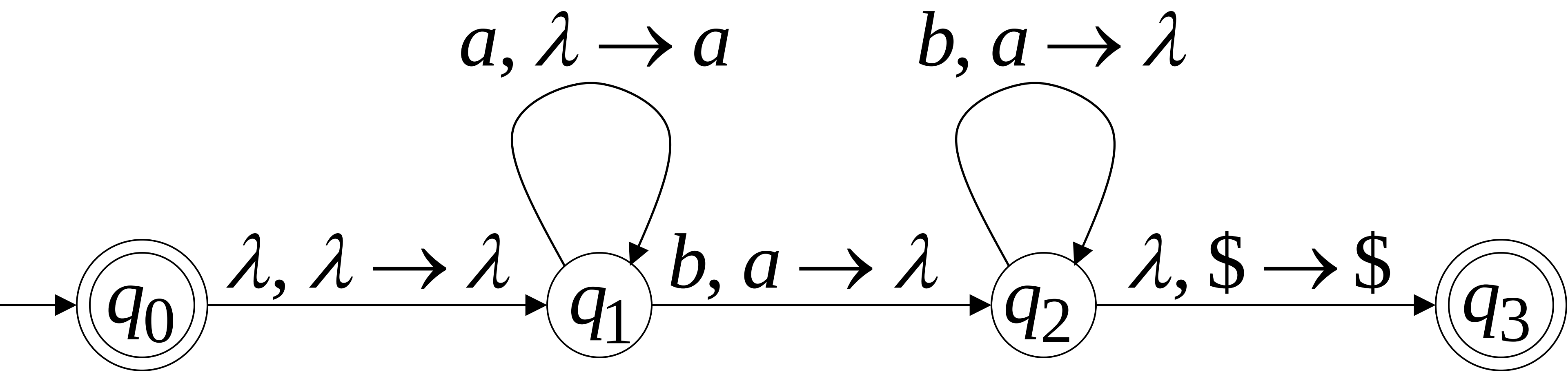
All the input is consumed

AND

The last state is a final state

At the end of the computation,
we do not care about the stack contents

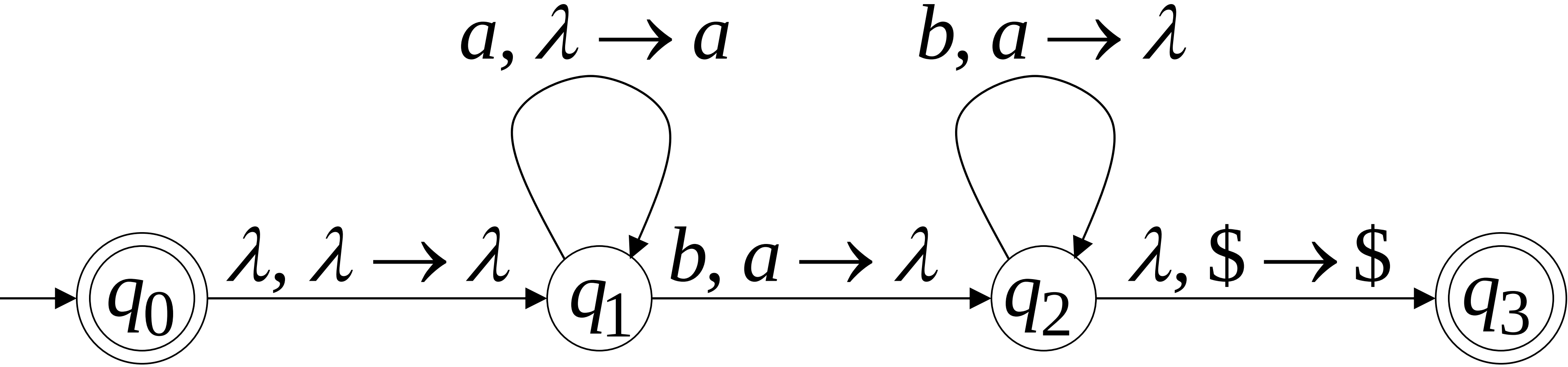
The input string $aaabbb$
is accepted by the NPDA:



In general,

$$L=\{a^n b^n : n \geq 0\}$$

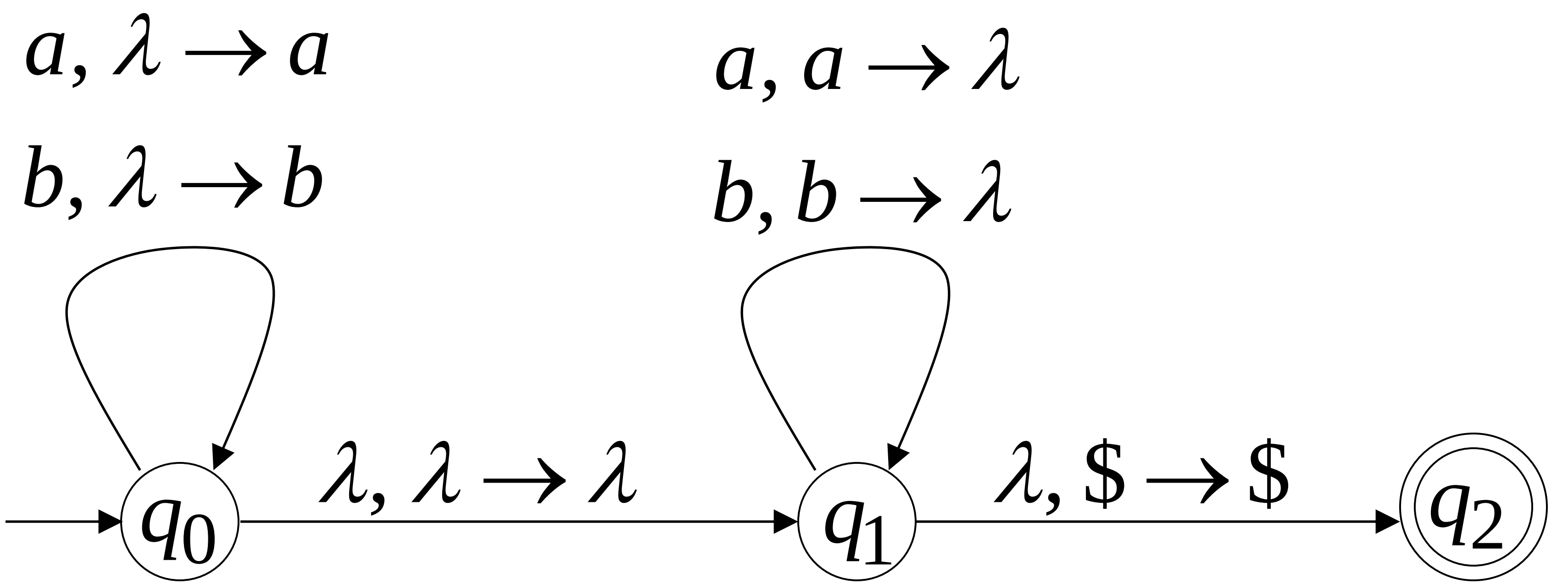
is the language accepted by the NPDA:



Another NPDA example

NPDA M

$$L(M) = \{ww^R\}$$



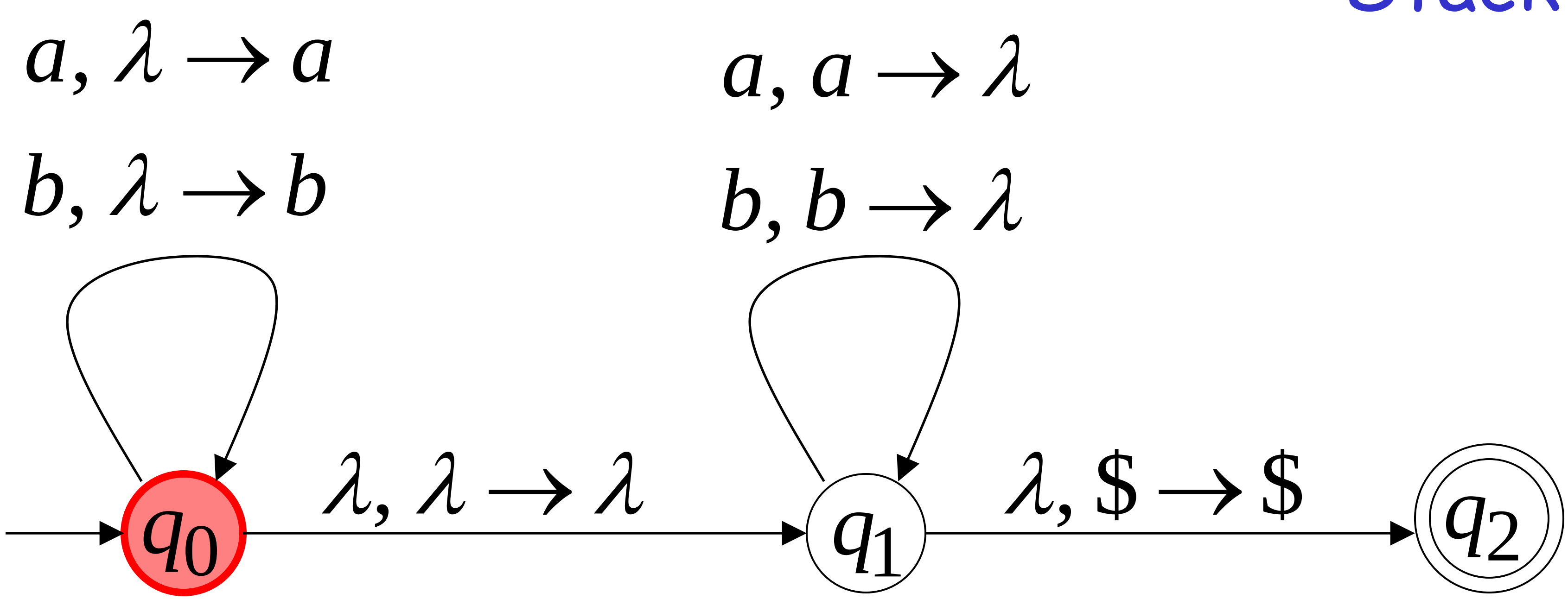
Execution Example: Time 0

Input

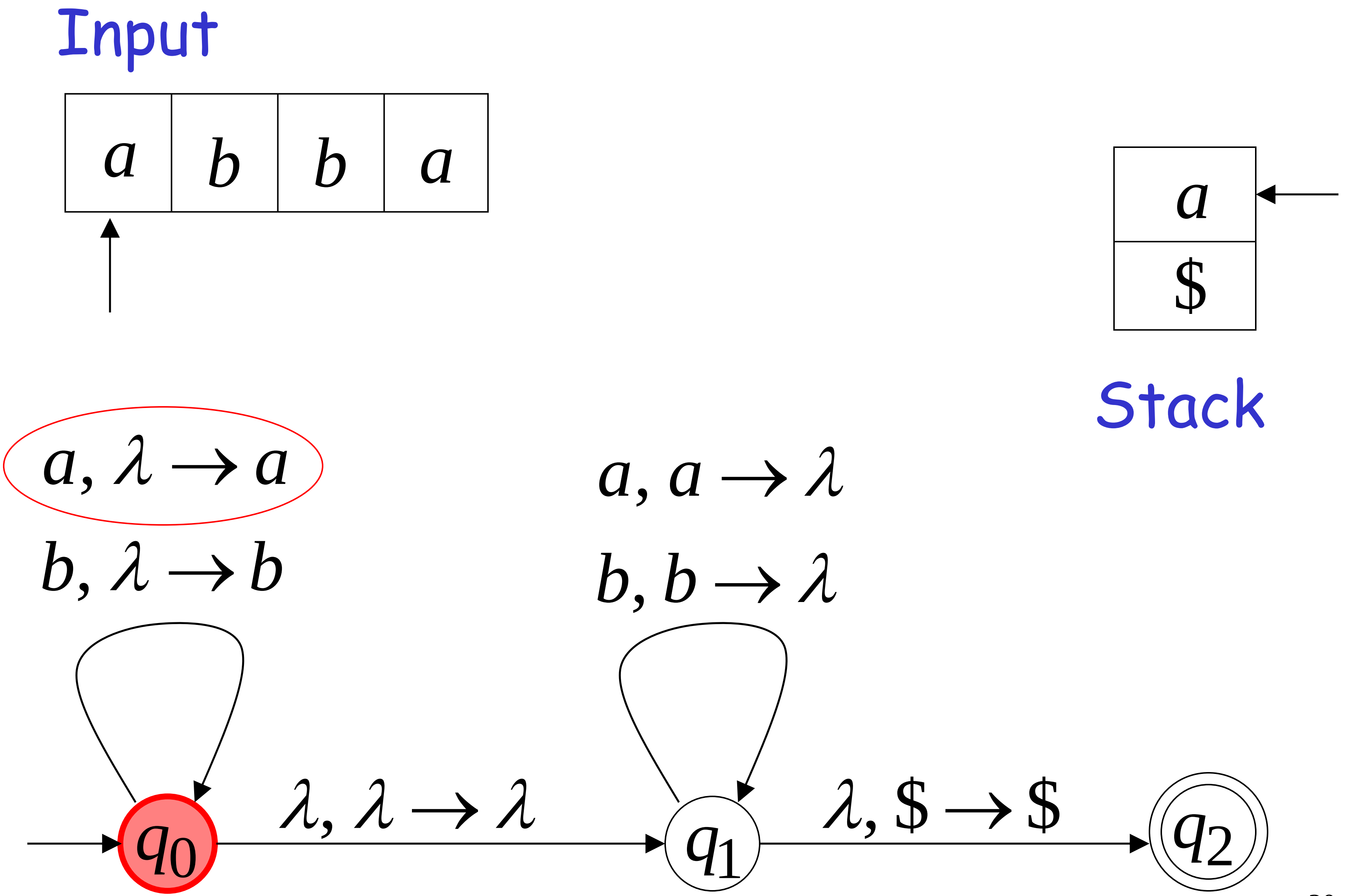
a	b	b	a

$\$$

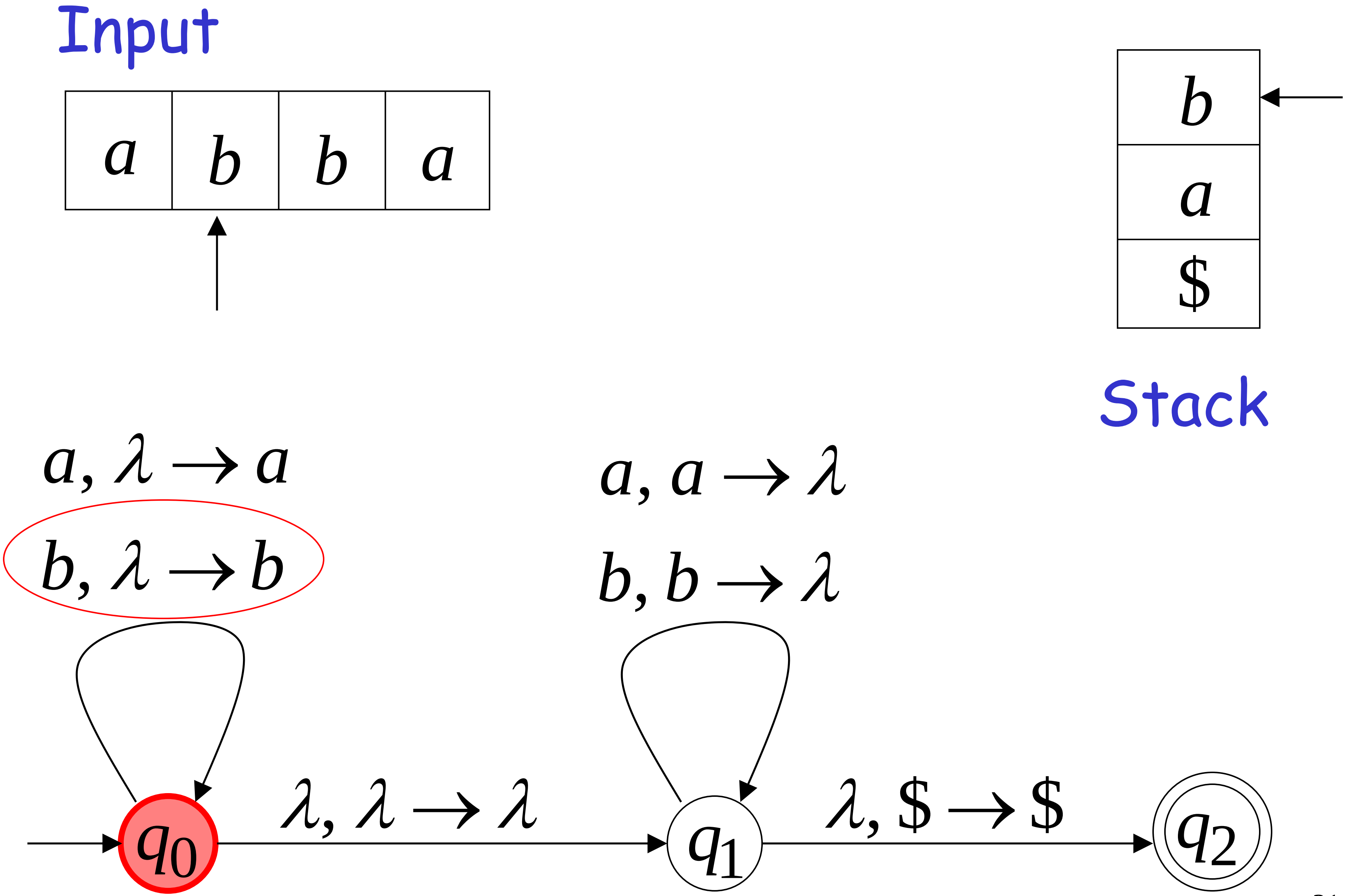
Stack



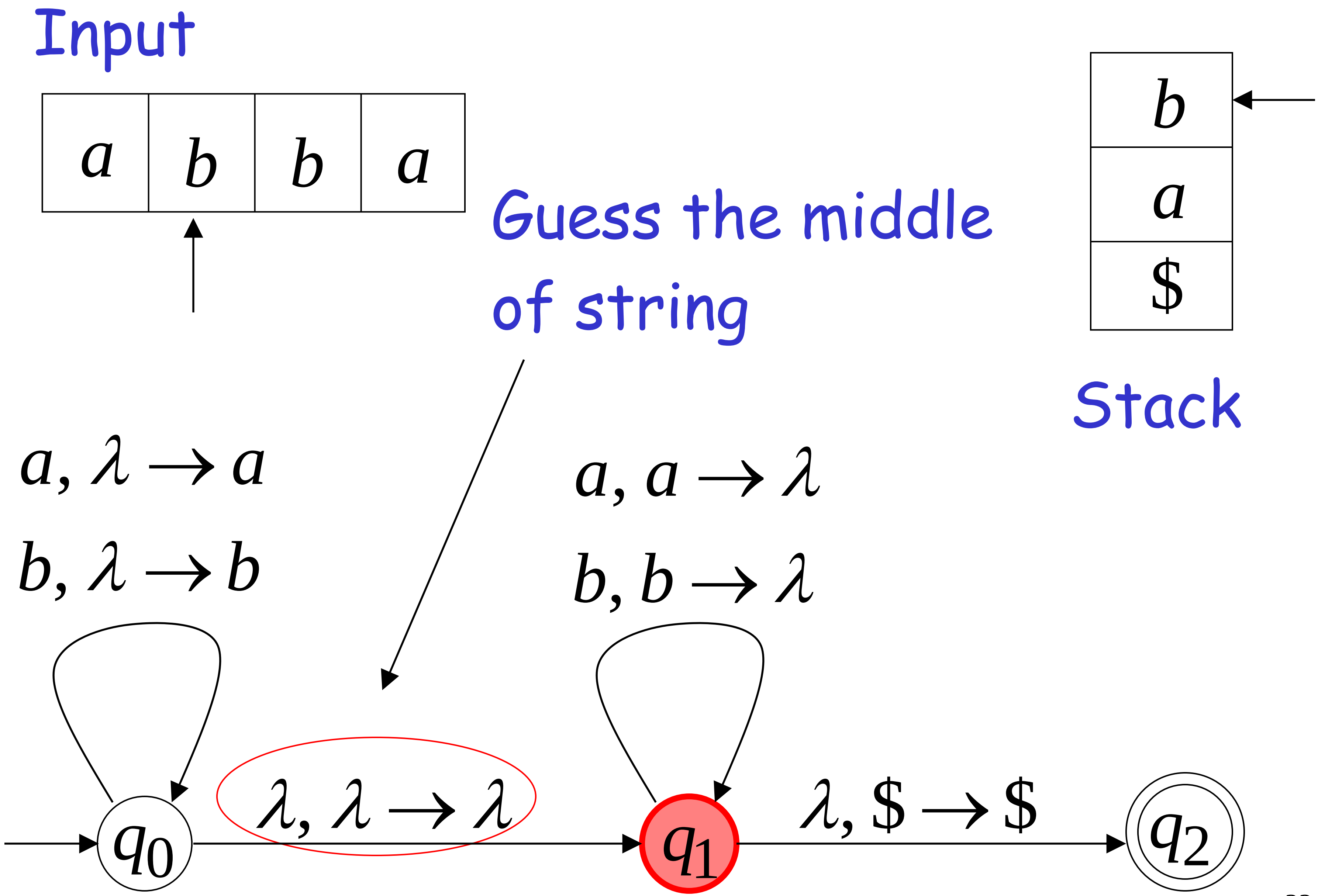
Time 1


Input
a b b a
a
$\$$
Stack
$a, \lambda \to a$
$b, \lambda \to b$
$a, a \to \lambda$
$b, b \to \lambda$
$\lambda, \lambda \to \lambda$
$\lambda, \$ \to \$$
q_0
q_1
q_2

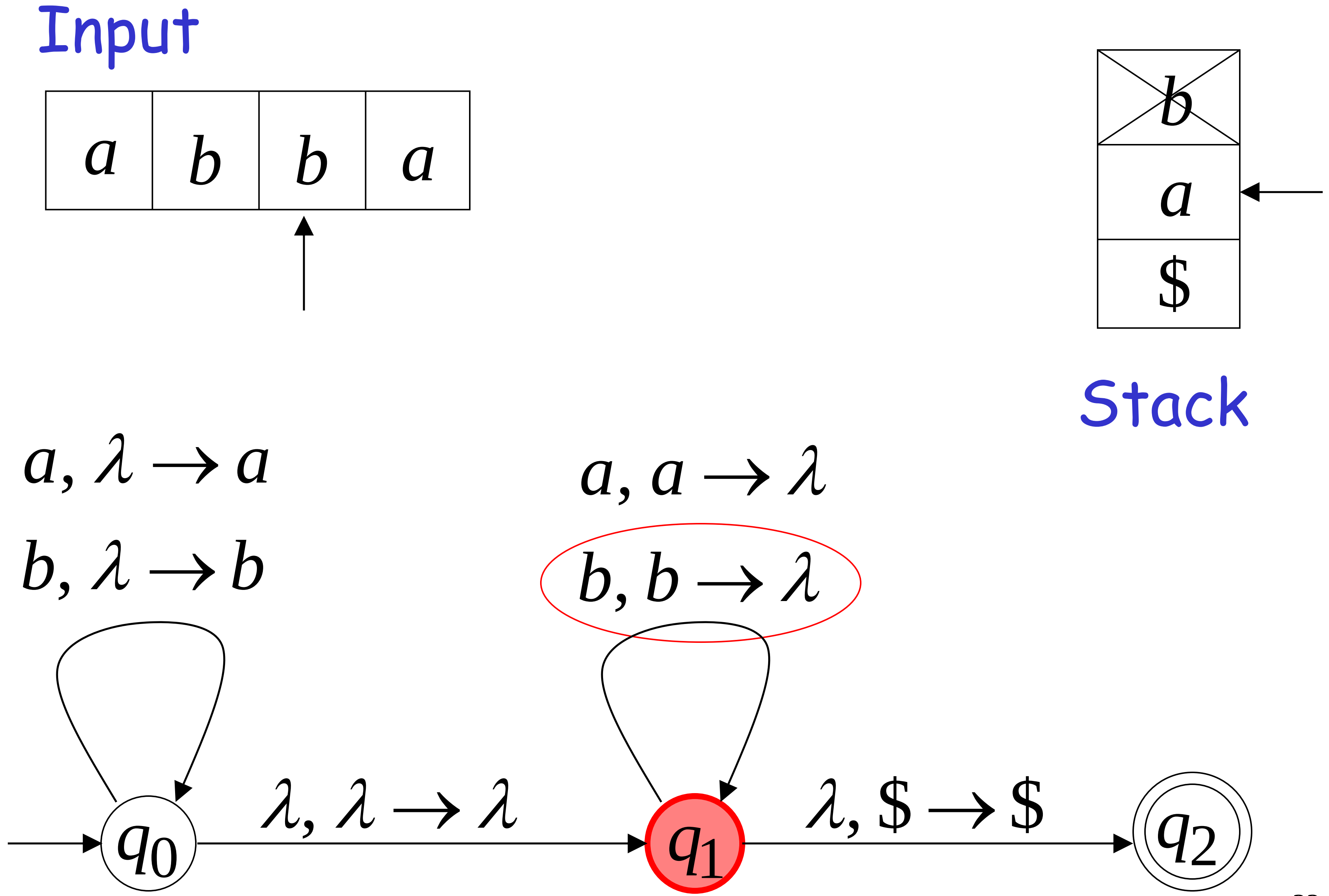
Time 2


Input
a b b a
b
a
$
Stack
$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

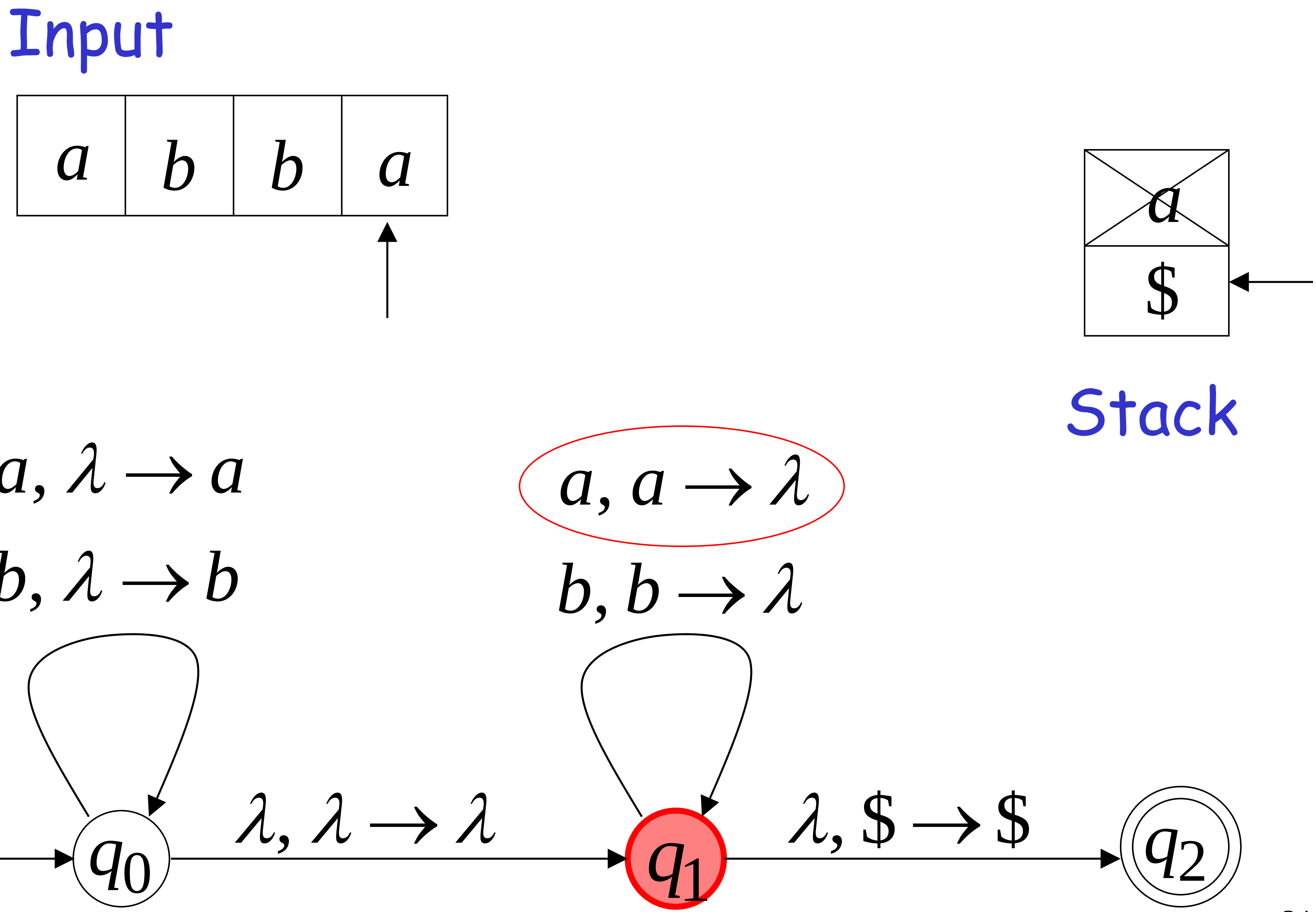
Time 3


Input
a b b a
Guess the middle
of string
b
a
$
Stack
$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Time 4

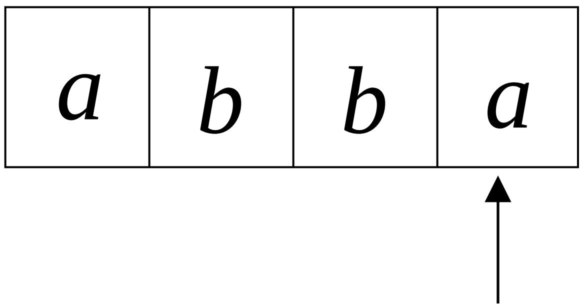

Input
a
b
b
a
$\cancel{b}$
a
$
Stack
$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Time 5

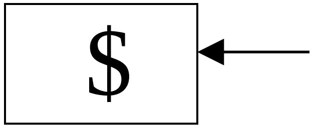

Input
a b b a
a
$
Stack
$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Time 6

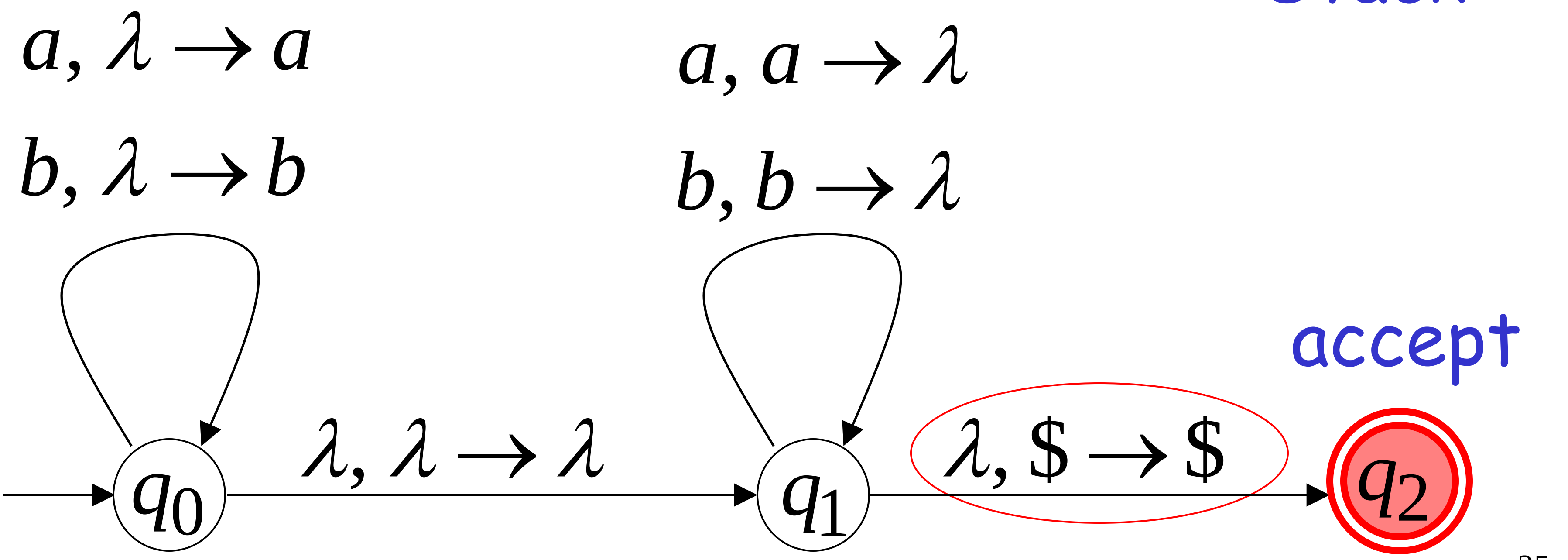
Input



a b b a

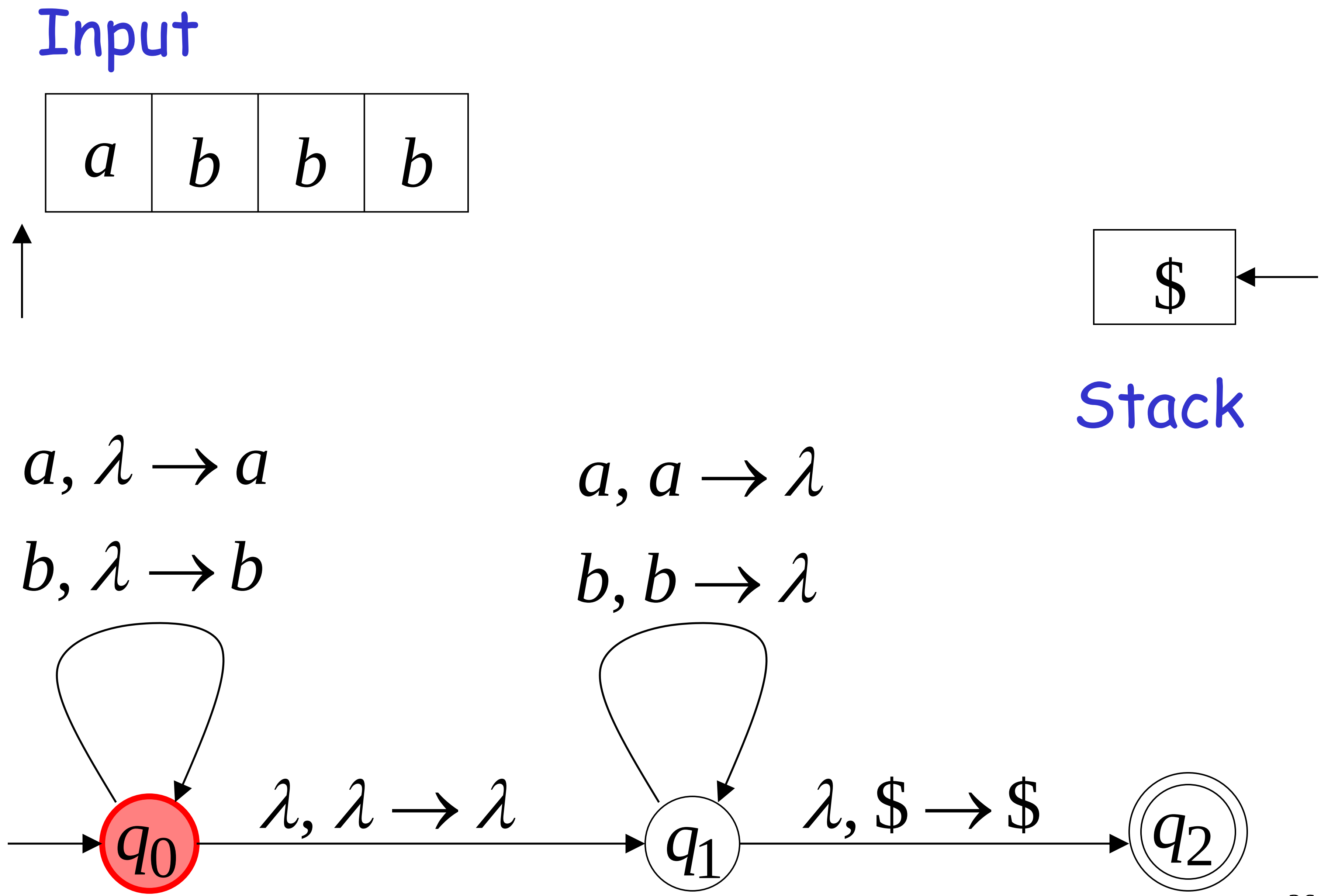


$
Stack

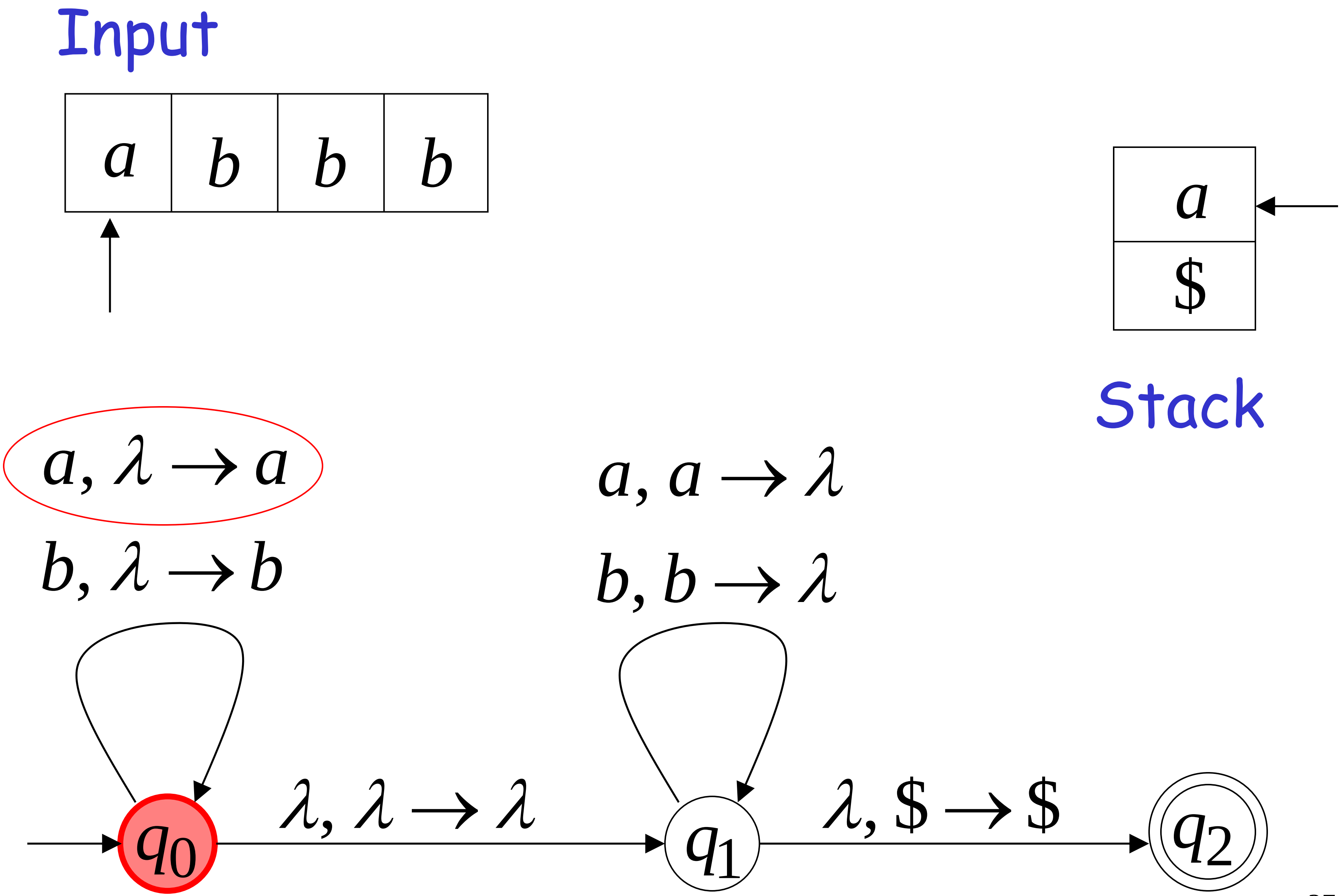


$a, \lambda \to a$
$b, \lambda \to b$
$a, a \to \lambda$
$b, b \to \lambda$
$\lambda, \lambda \to \lambda$
$\lambda, \$ \to \$$
q_0
q_1
q_2
accept

Rejection Example: Time 0

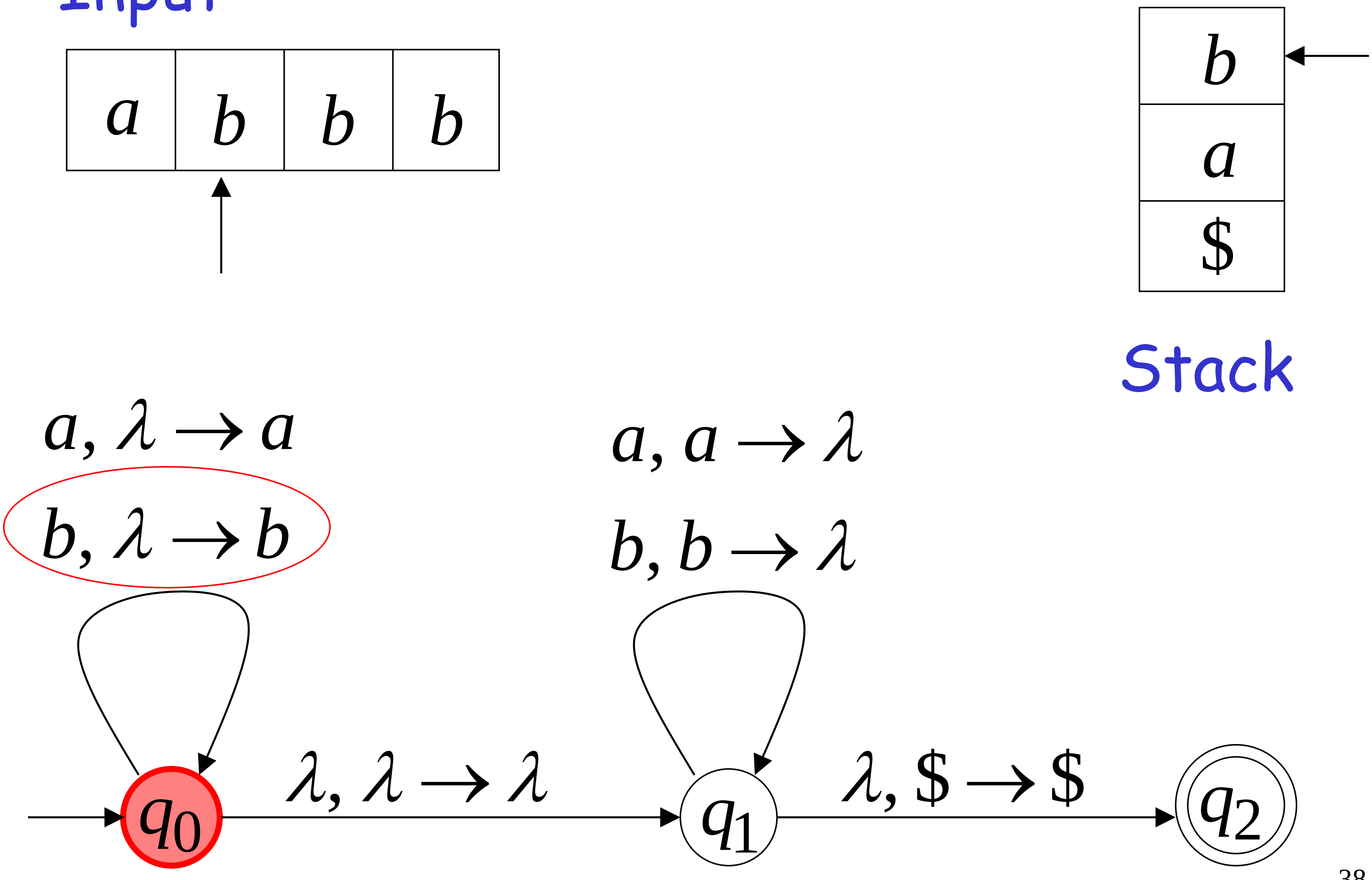


Time 1


Input
a b b b
a
$
Stack
$a, \lambda \to a$
$b, \lambda \to b$
$a, a \to \lambda$
$b, b \to \lambda$
$\lambda, \lambda \to \lambda$
$\lambda, \$ \to \$$
q_0
q_1
q_2

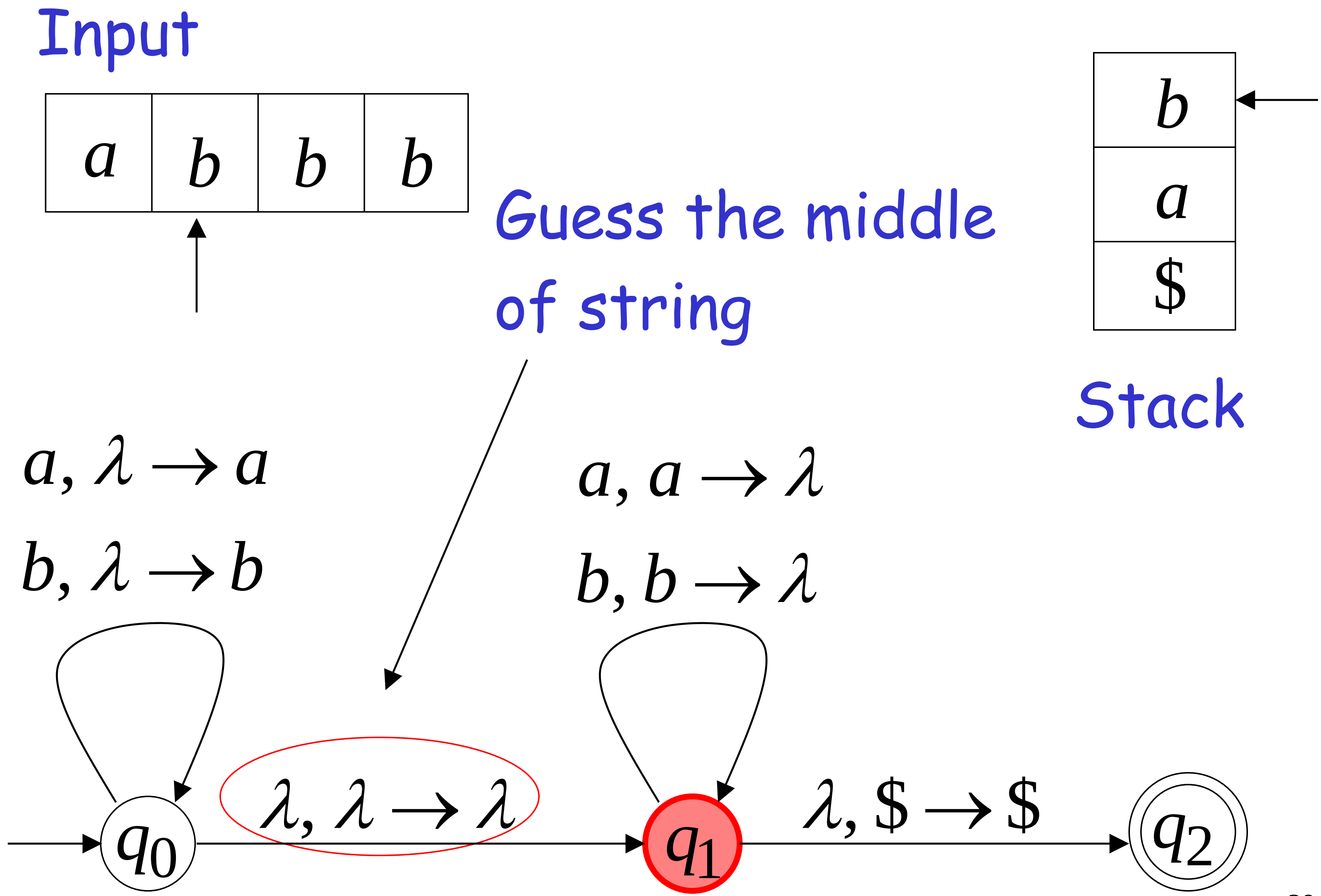
Time 2

Input



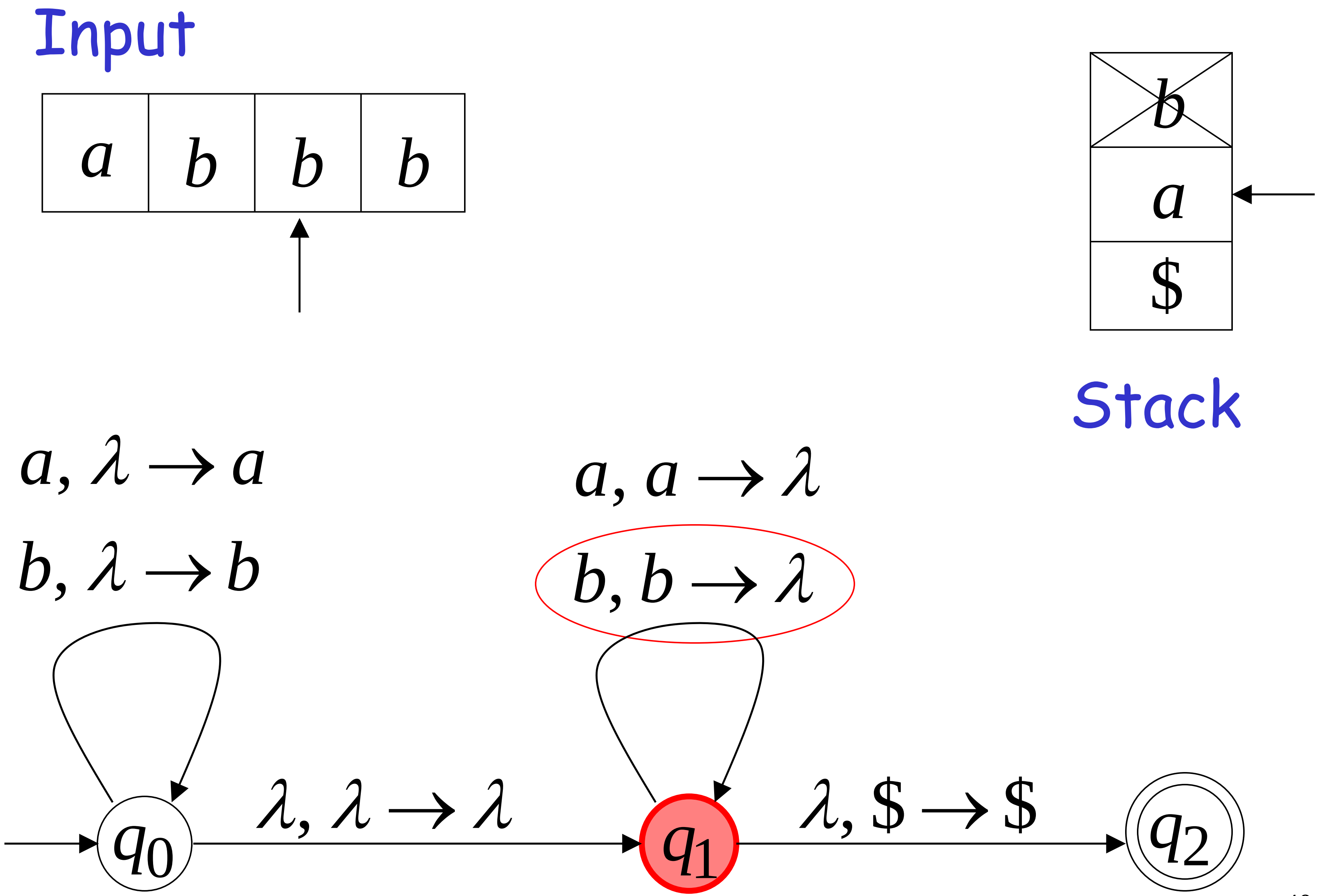
a
b
b
b
b
a
$\$$
Stack
$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Time 3



Input
a b b b
Guess the middle of string
b
a
$
Stack
$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

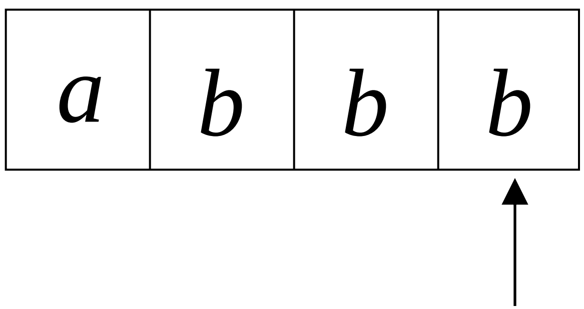
Time 4



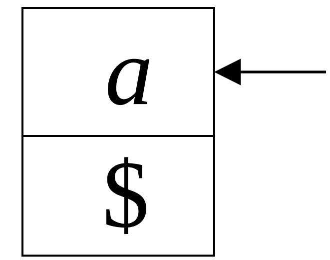
Time 5

There is no possible transition.

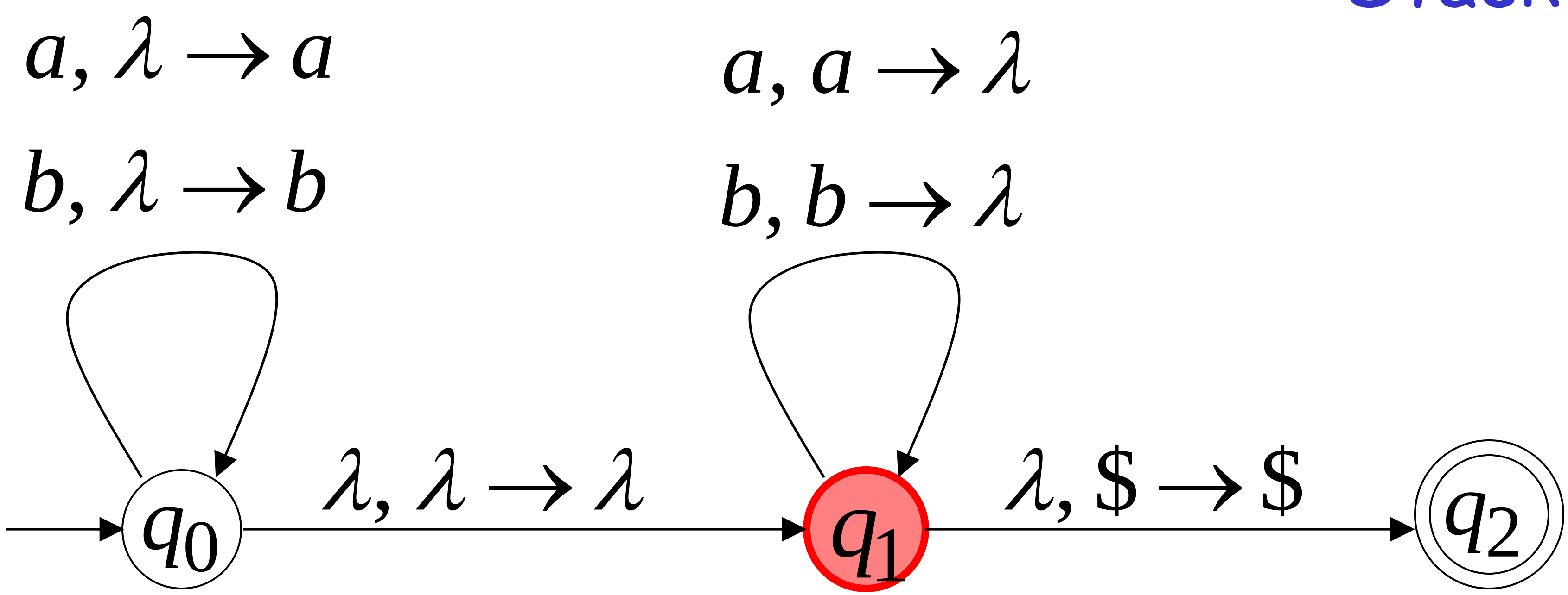
Input



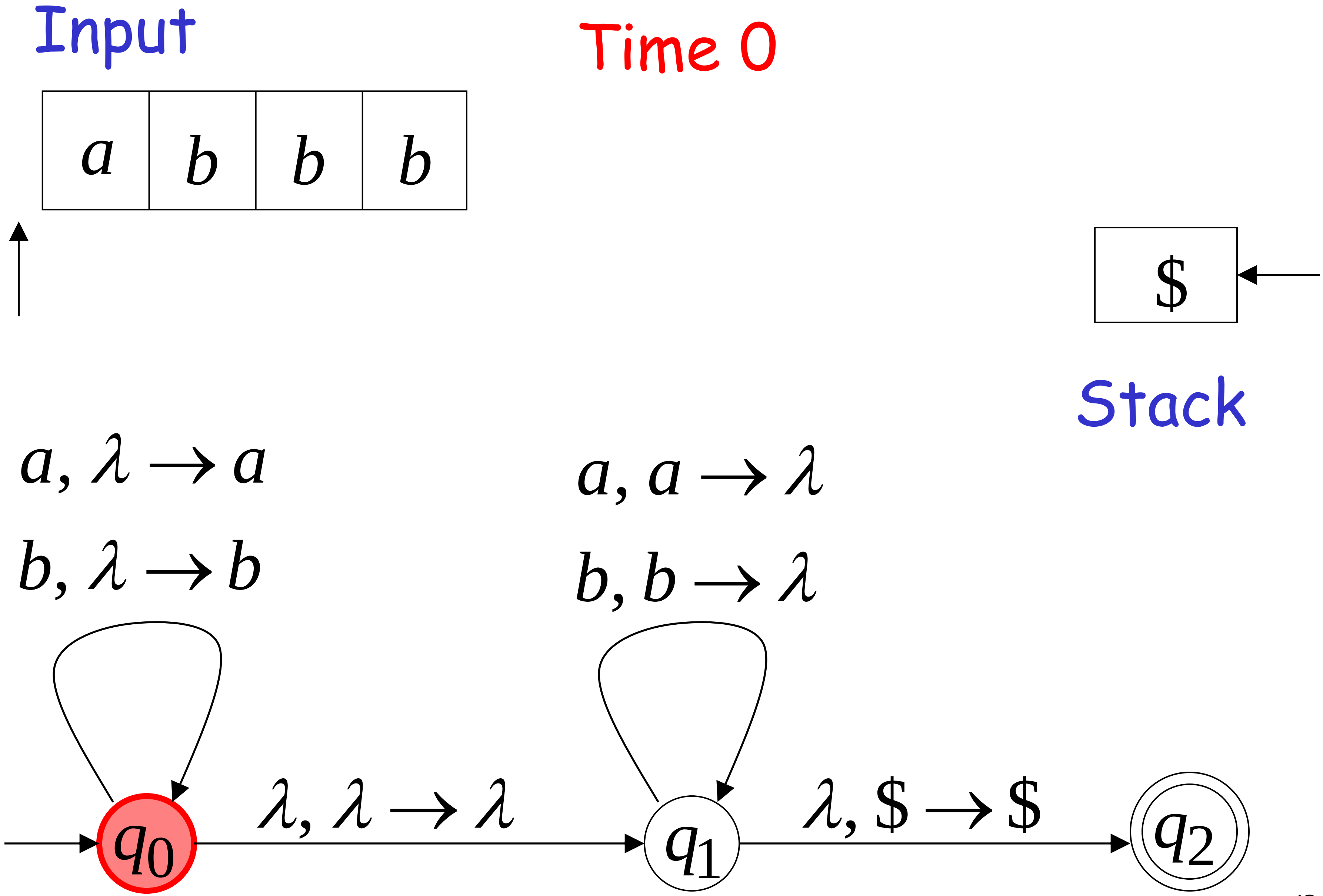
Input is not consumed



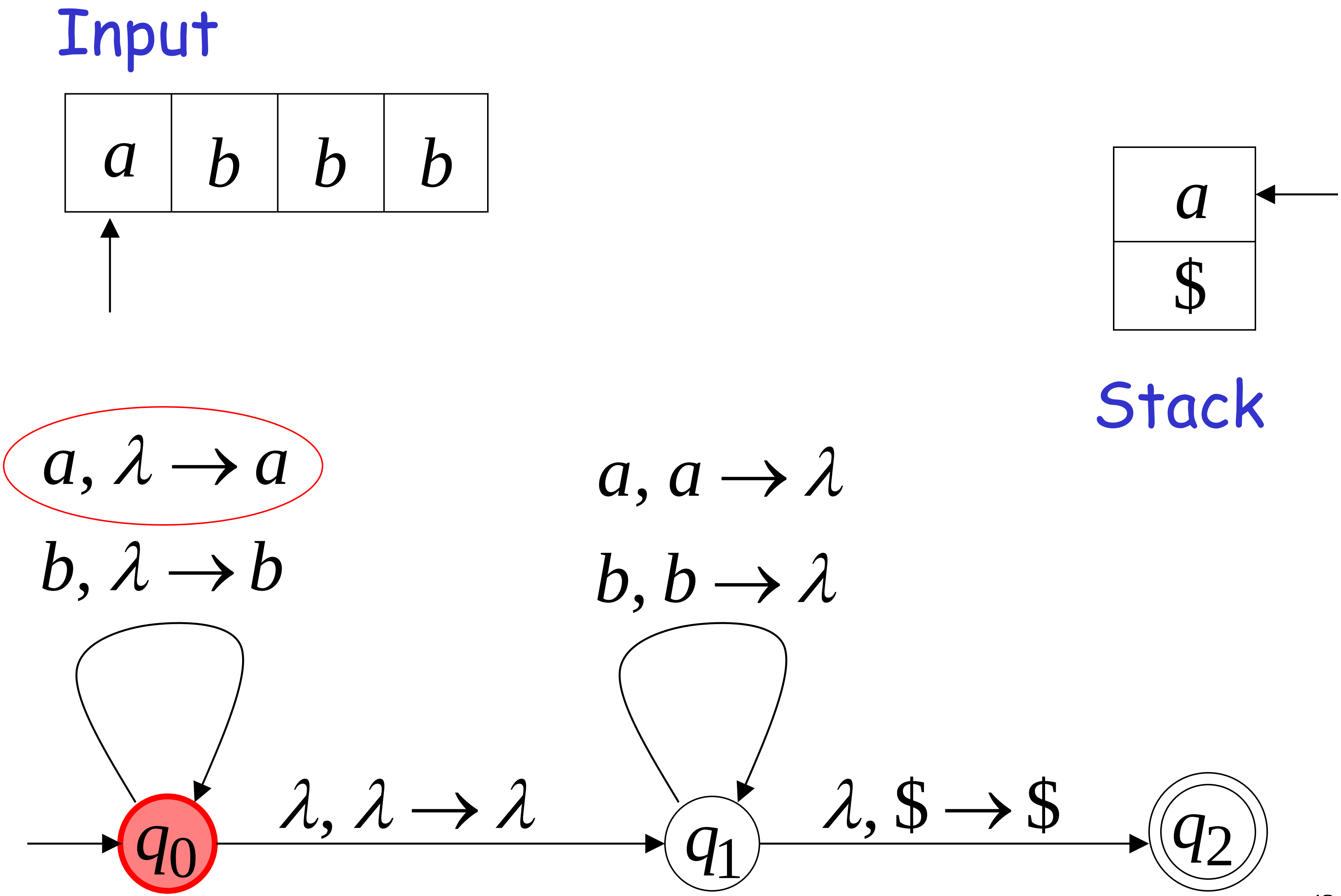
Stack



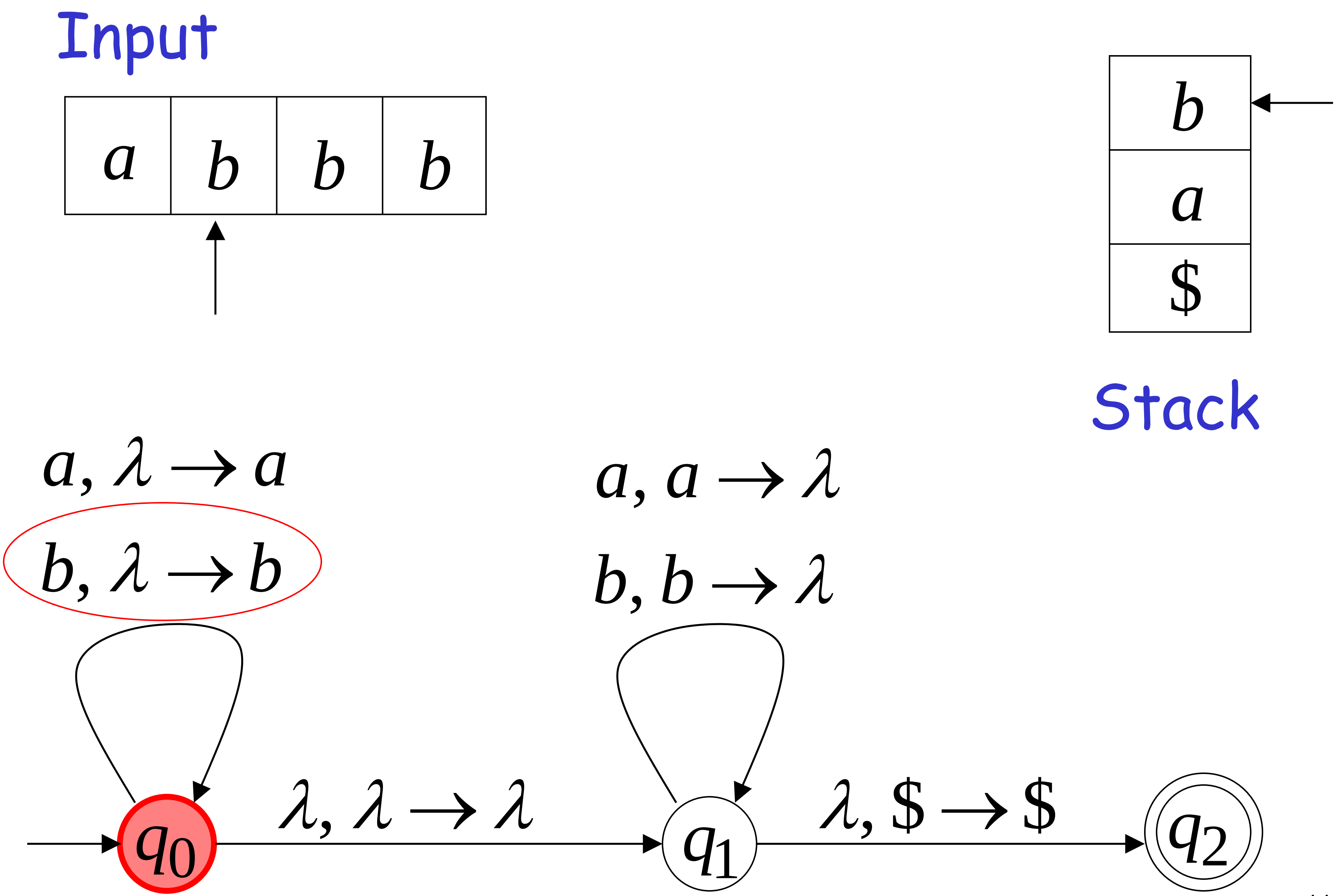
Another computation on same string:



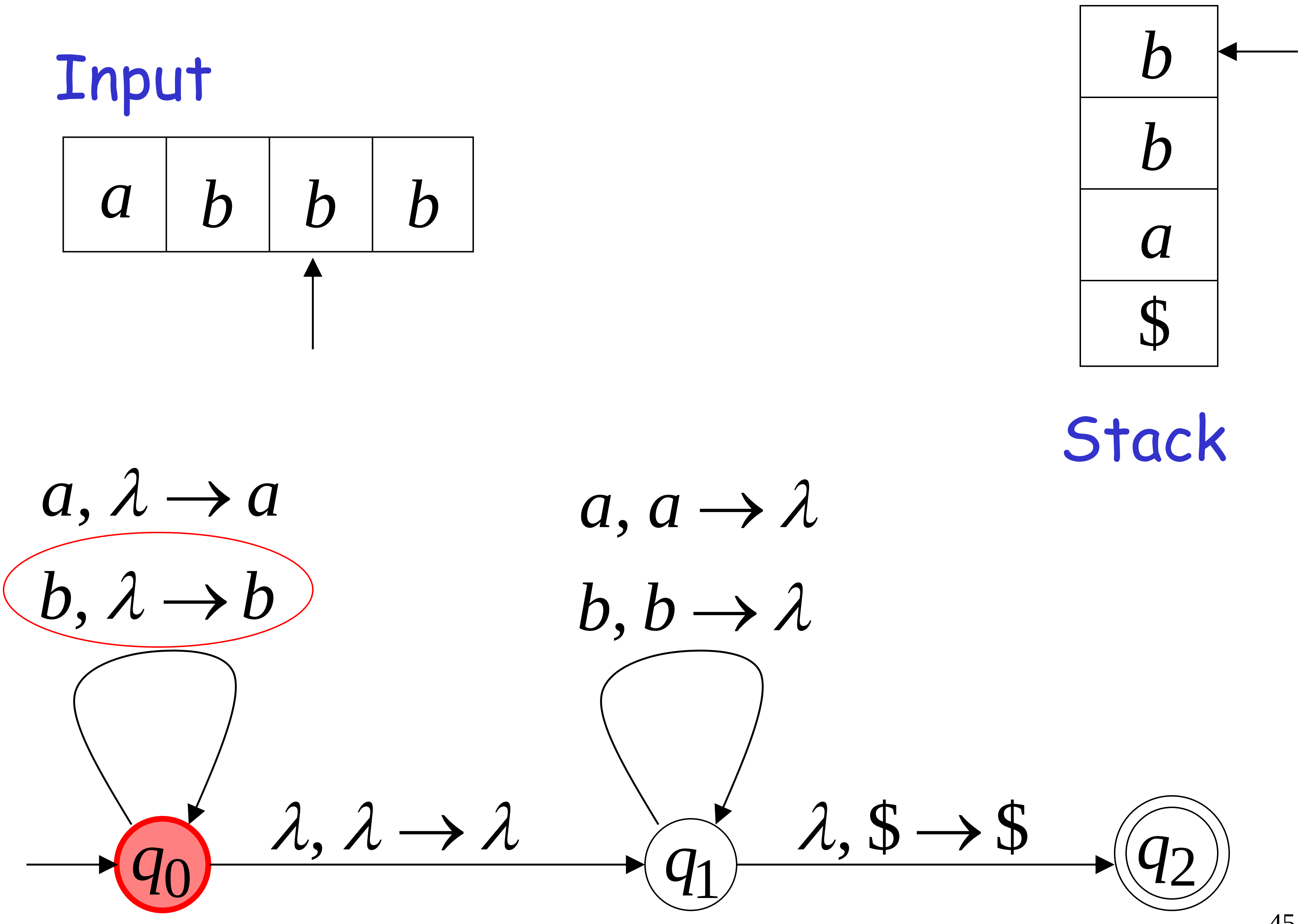
Time 1



Time 2

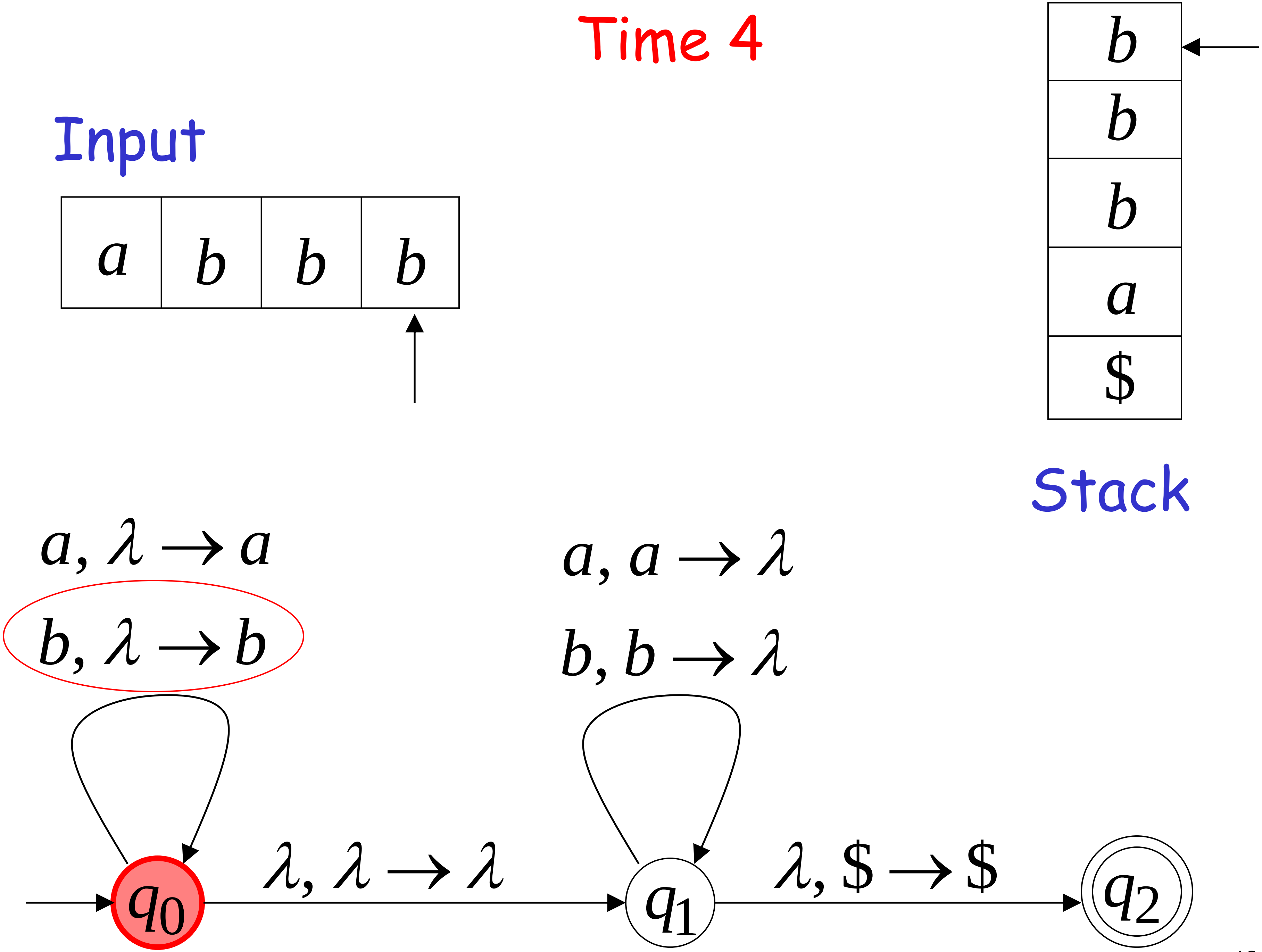

Input
a b b b
b
a
$\$$
Stack
$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Time 3



Input
a
b
b
b
b
b
a
$\$$
Stack
$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Time 4



Input
a b b b
b
b
b
a
$\$$
Stack
$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Time 5

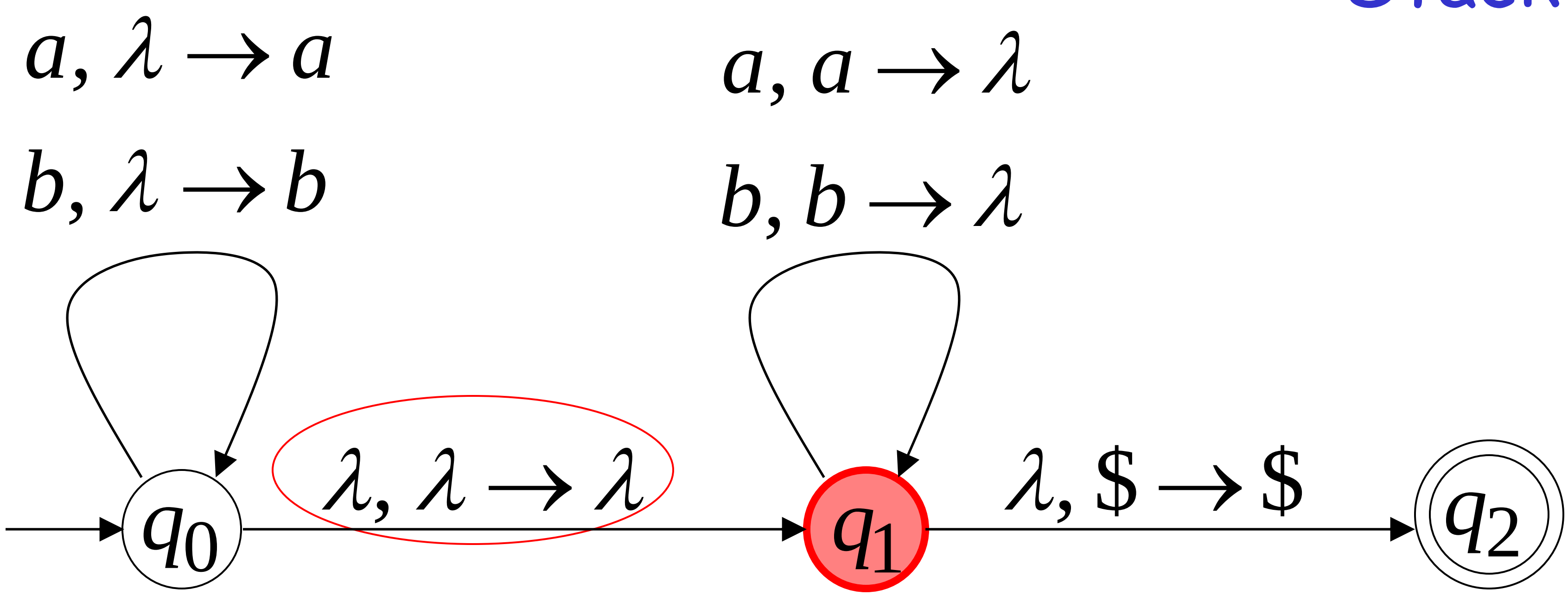
Input

a	b	b	b

No final state is reached

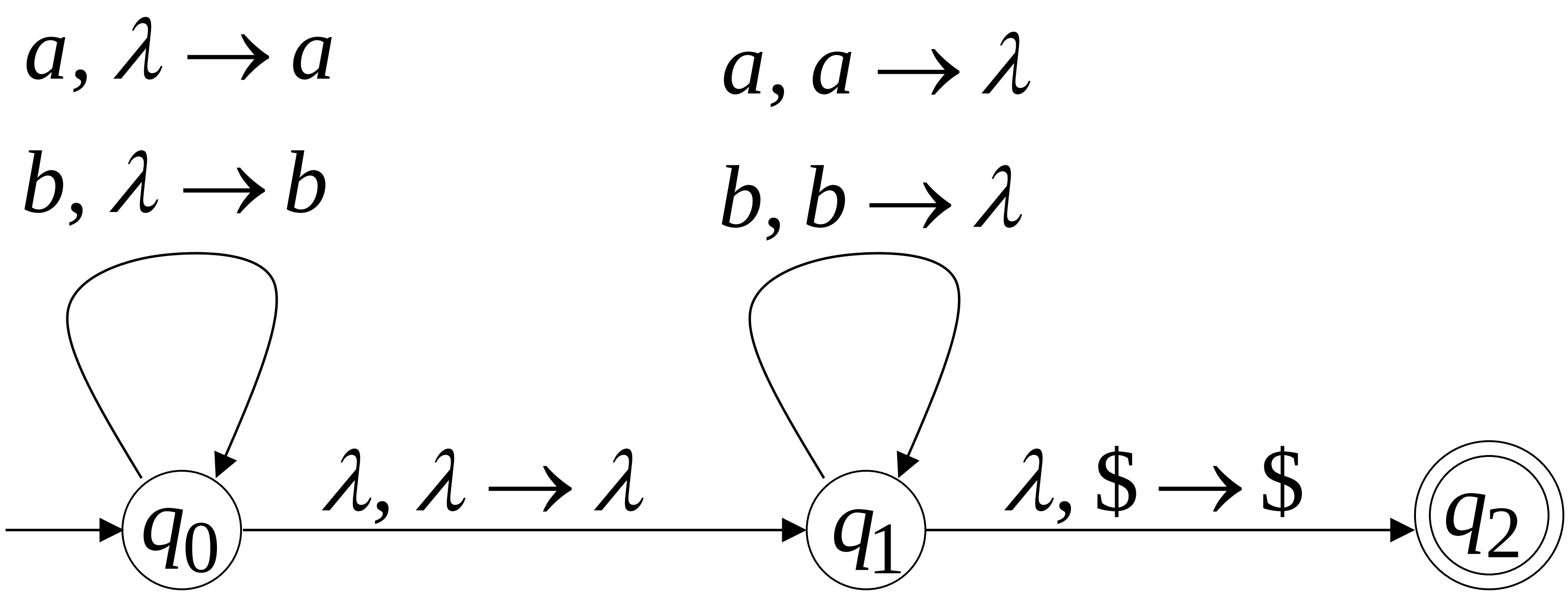
b
b
b
a
$\$$

Stack



There is no computation that accepts string $abbb$

$$abbb \notin L(M)$$



A string is rejected if there is no computation such that:

All the input is consumed

AND

The last state is a final state

At the end of the computation, we do not care about the stack contents

In other words, a string is rejected if in every computation with this string:

The input cannot be consumed

OR

The input is consumed and the last state is not a final state

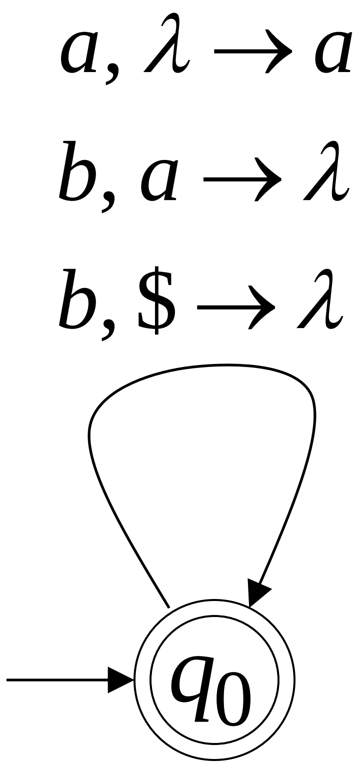
OR

The stack head moves below the bottom of the stack

Another NPDA example

NPDA M

$$L(M)=\{w:\ w\in\{a,b\}^*,\ n_a\geq n_b\ \ \textit{for any prefix of}\ \ w\}$$



Execution Example: Time 0

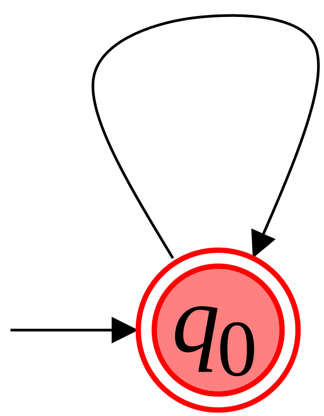
Input

a	a	b

$$a, \lambda \rightarrow a$$

$$b, a \rightarrow \lambda$$

$$b, \$ \rightarrow \lambda$$



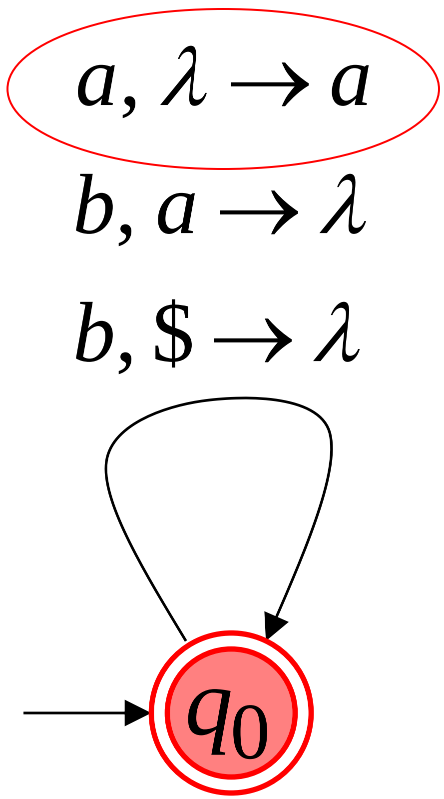
$\$$

Stack

Time 1

Input

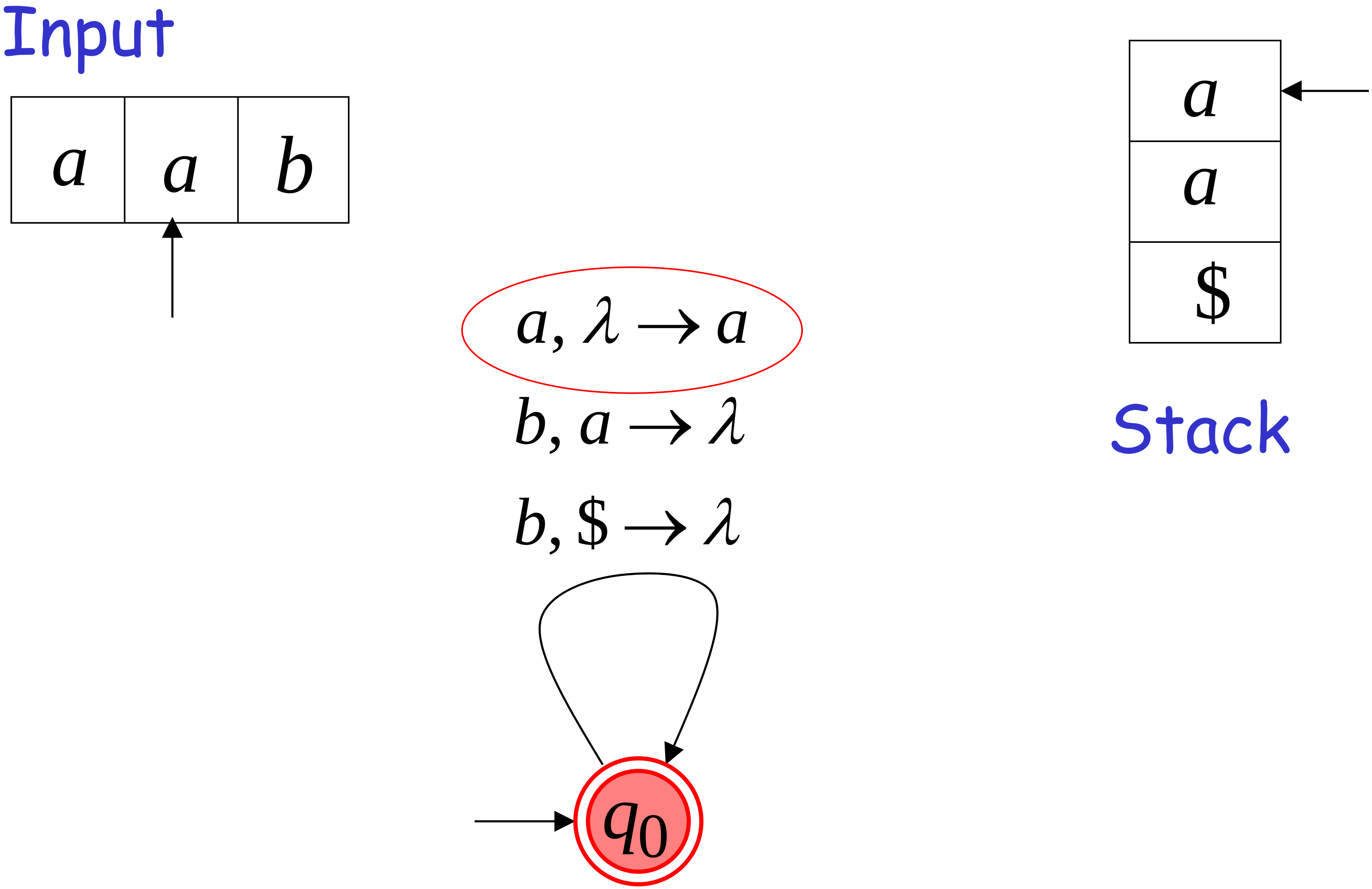
a	a	b



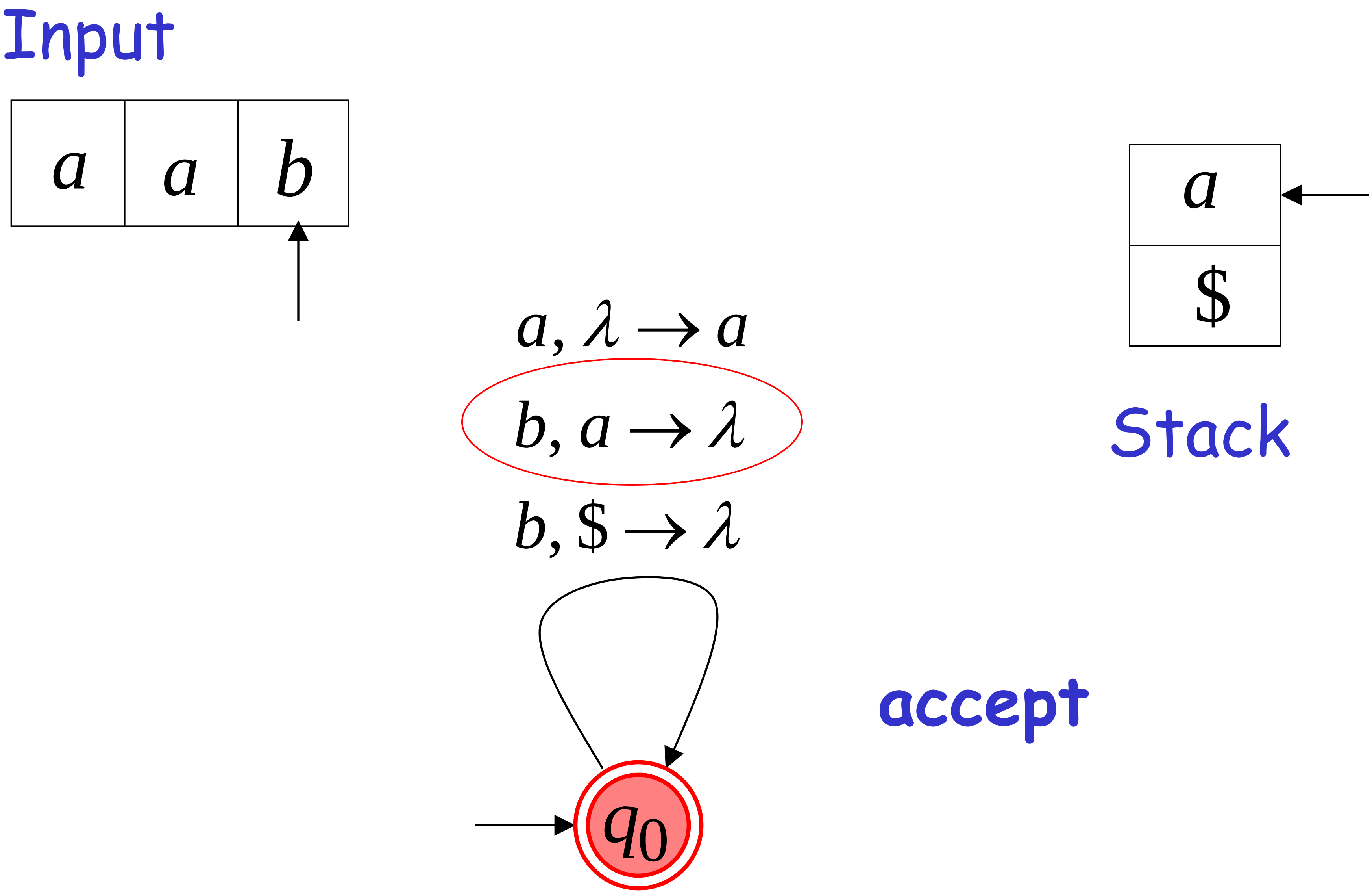
a
$\$$

Stack

Time 2


Input
a
a
b
$a, \lambda \rightarrow a$
$b, a \rightarrow \lambda$
$b, \$ \rightarrow \lambda$
q_0
a
a
$
Stack

Time 3


Input
a a b
$a, \lambda \to a$
$b, a \to \lambda$
$b, \$ \to \lambda$
q_0
Stack
a
$
accept

Rejection example:

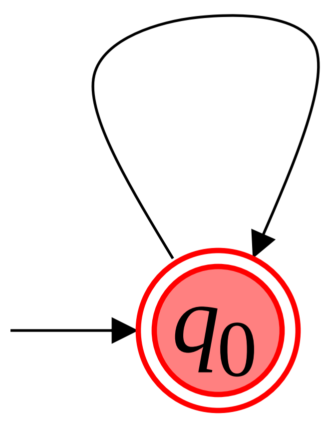
Time 0

Input

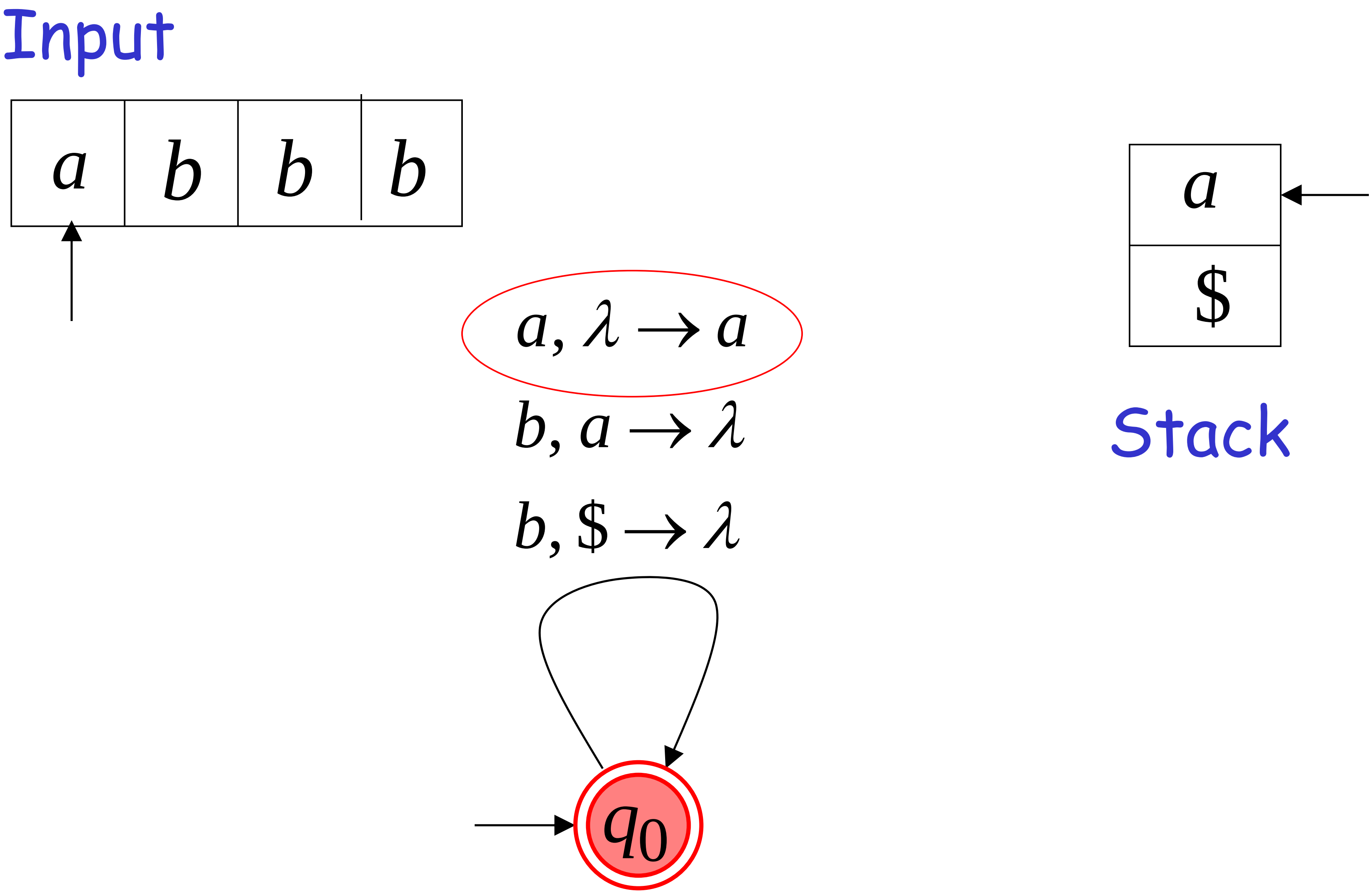
a	b	b	b

$\$$

Stack



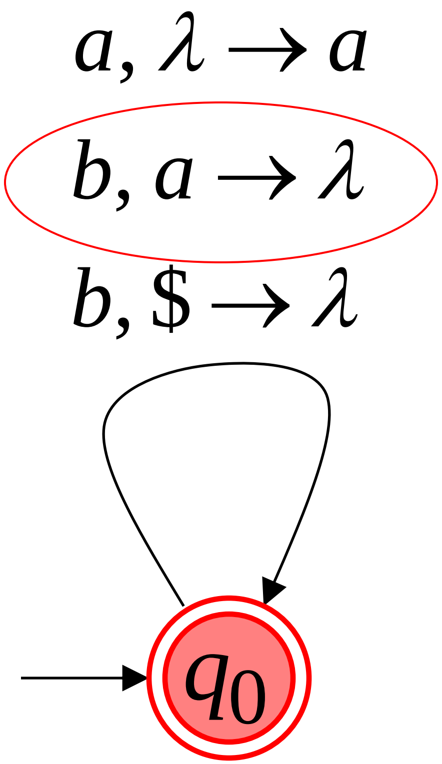
Time 1



Time 2

Input

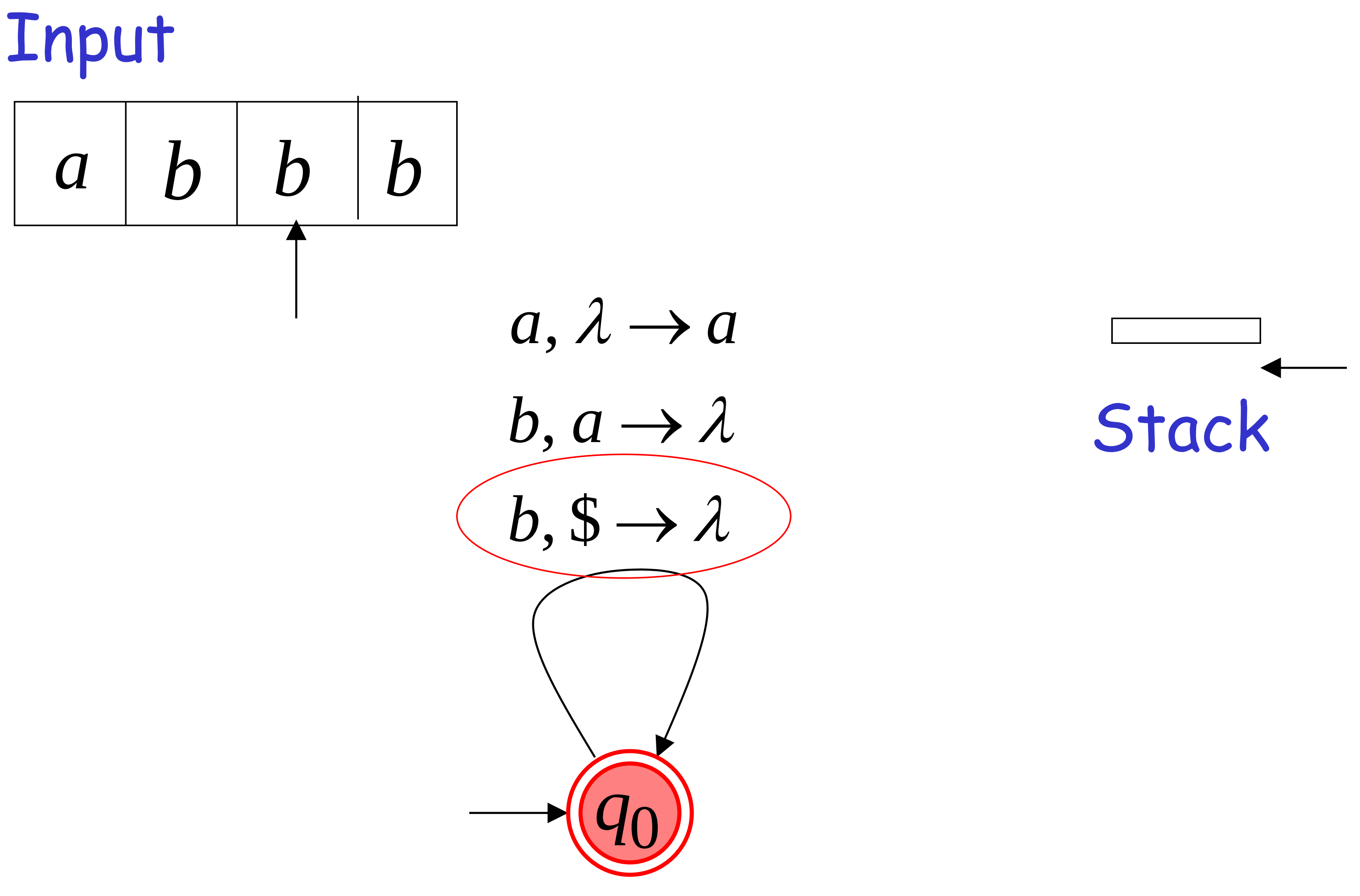
a	b	b	b



$\$$

Stack

Time 3



Time 4

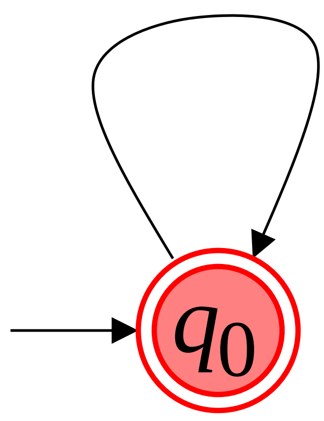
Input

a	b	b	b

$$a, \lambda \rightarrow a$$

$$b, a \rightarrow \lambda$$

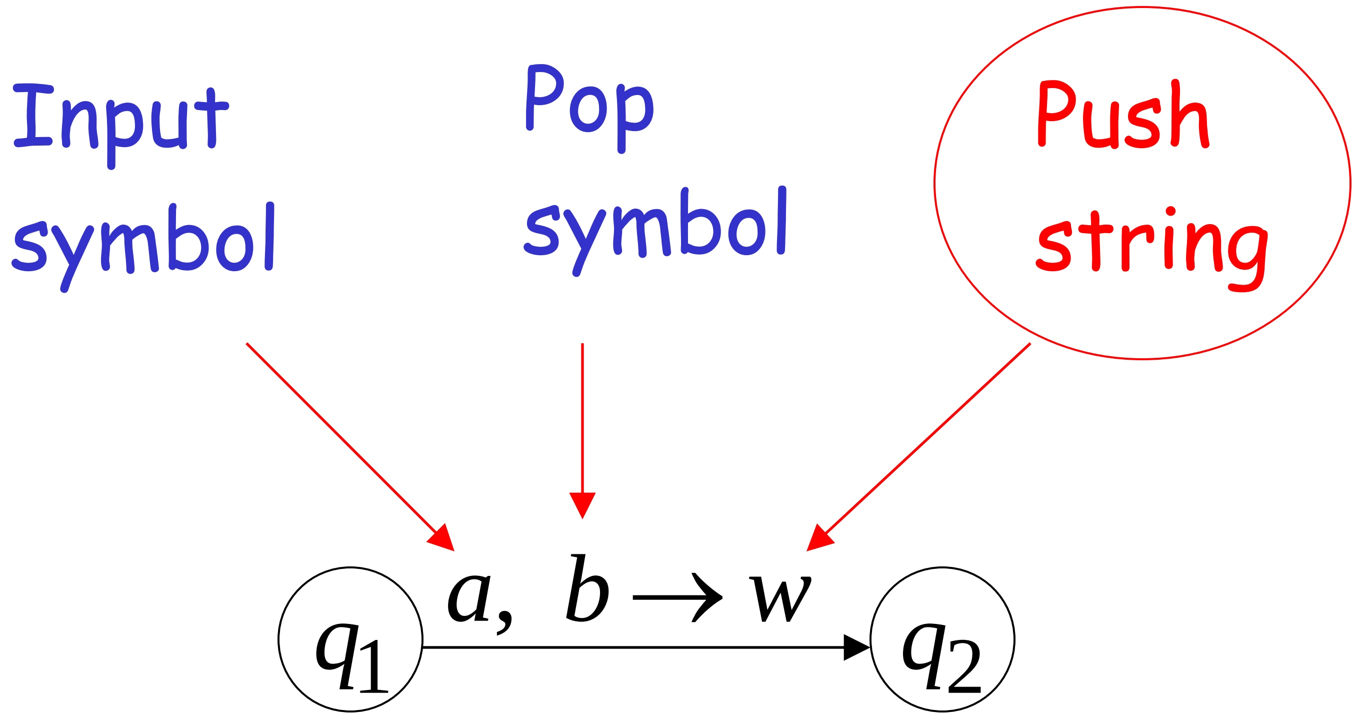
$$b, \$ \rightarrow \lambda$$



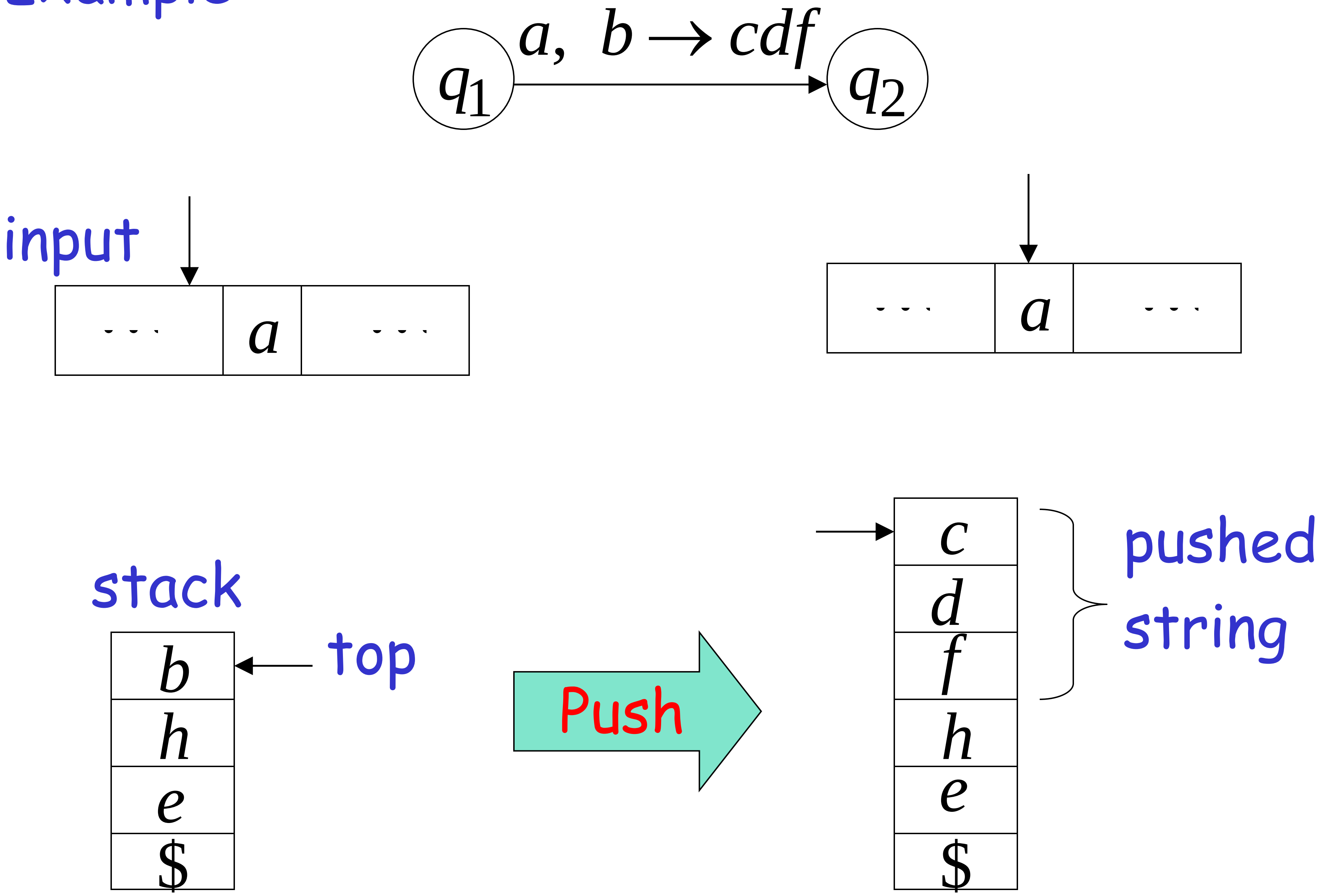
Stack

Halt and Reject

Pushing Strings



Example:

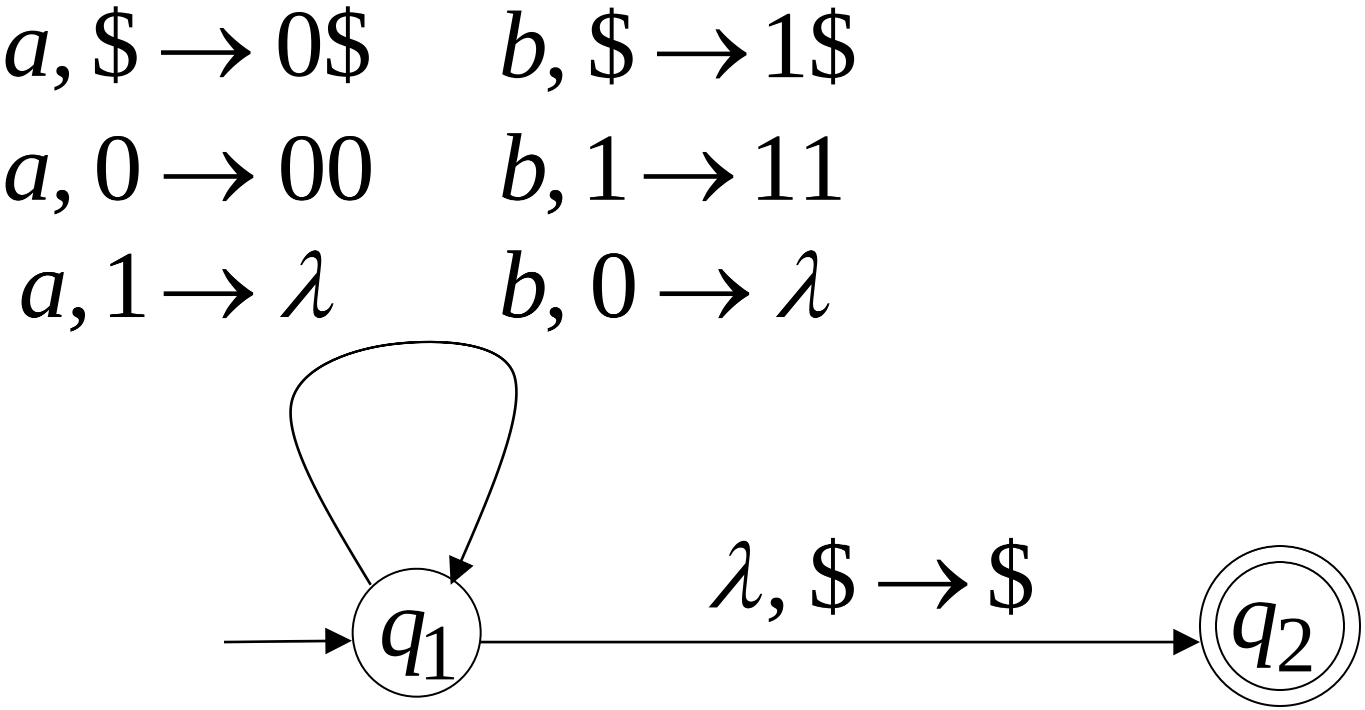


$a, \ b \rightarrow cdf$
q_1
q_2
input
a
a
stack
top
b
h
e
$\$$
Push
c
d
f
h
e
$\$$
pushed string

Another NPDA example

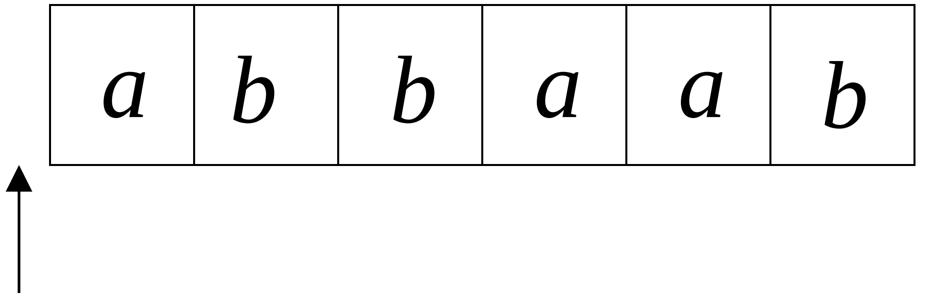
NPDA M

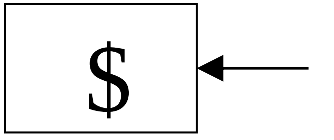
$$L(M) = \{w: \ n_a = n_b\}$$



Execution Example: Time 0

Input





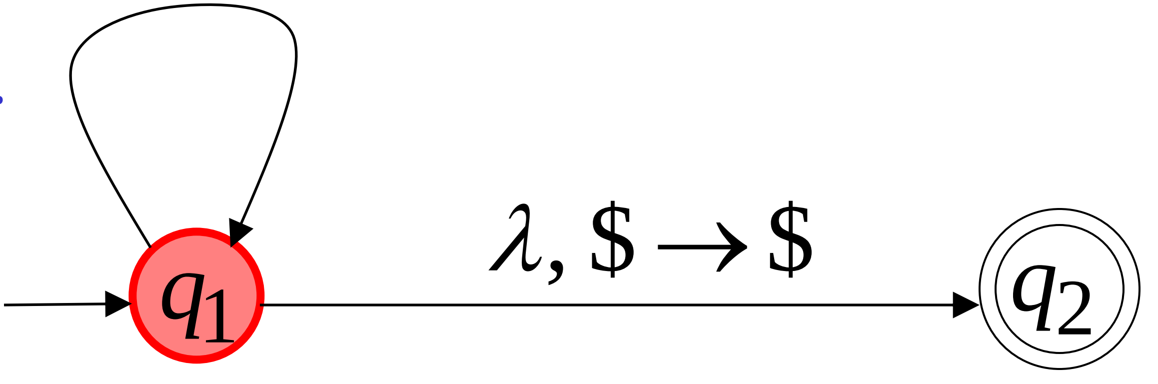
Stack

$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$

$$a, 1 \rightarrow \lambda \qquad b, 0 \rightarrow \lambda$$

current state



Time 1

Input

a	b	b	a	a	b

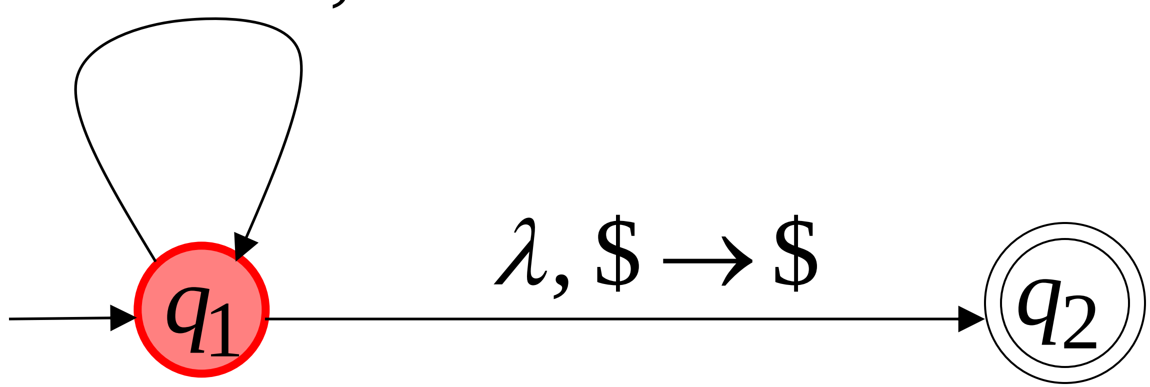
0
\$

Stack

$a, \$ \rightarrow 0\$$ $\quad$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $\quad$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda$ $\quad$ $b, 0 \rightarrow \lambda$



Time 3

Input

a	b	b	b	a	a

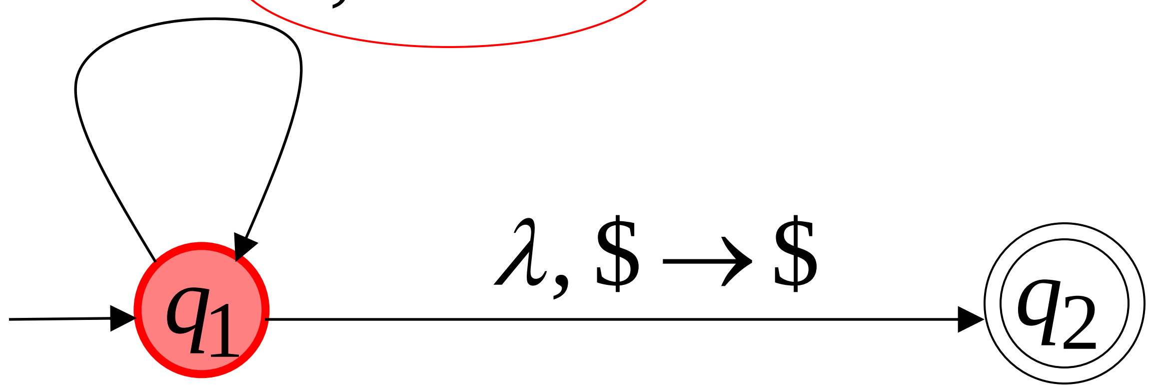
$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$

$$a, 1 \rightarrow \lambda \qquad b, 0 \rightarrow \lambda$$

$\cancel{0}$
$\$$

Stack



Time 4

Input

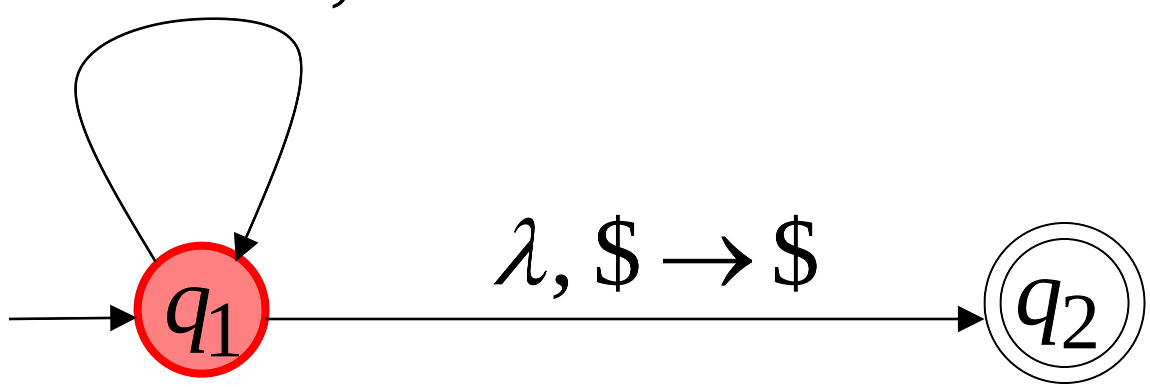
a	b	b	b	a	a

Stack
1
$

$a, \$ \rightarrow 0\$$ $\quad$ $b, \$ \rightarrow 1\$$

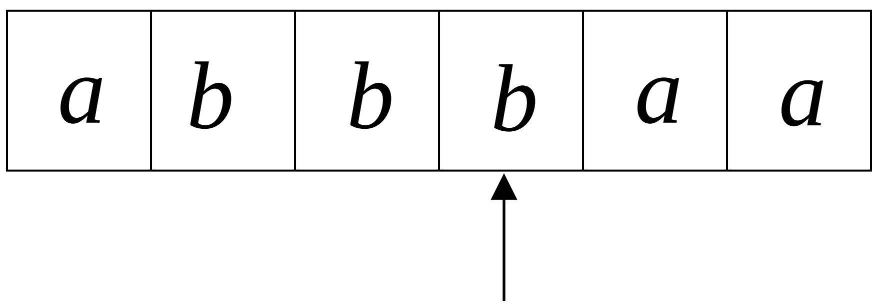
$a, 0 \rightarrow 00$ $\quad$ $b, 1 \rightarrow 11$

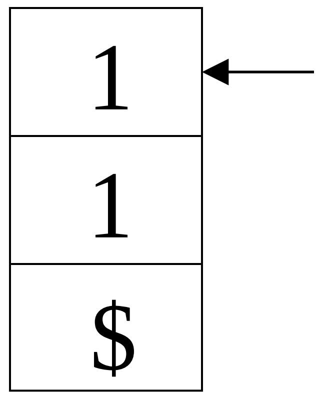
$a, 1 \rightarrow \lambda$ $\quad$ $b, 0 \rightarrow \lambda$



Time 5

Input



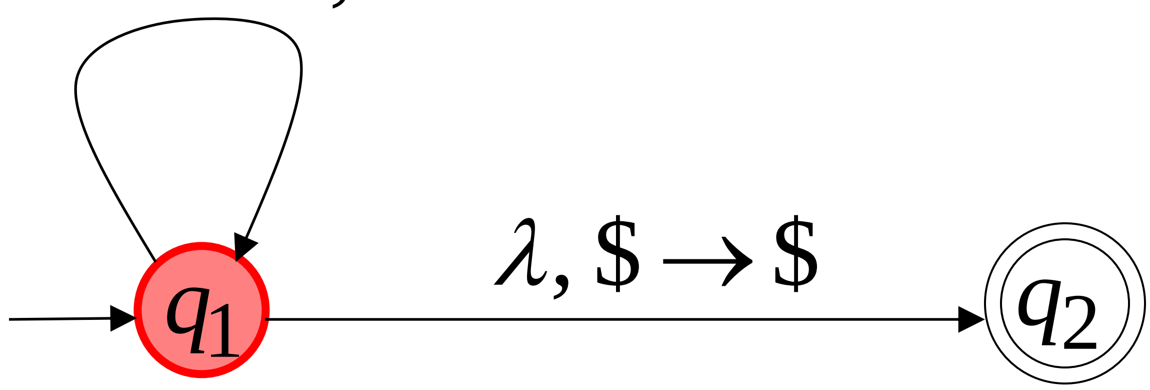


Stack

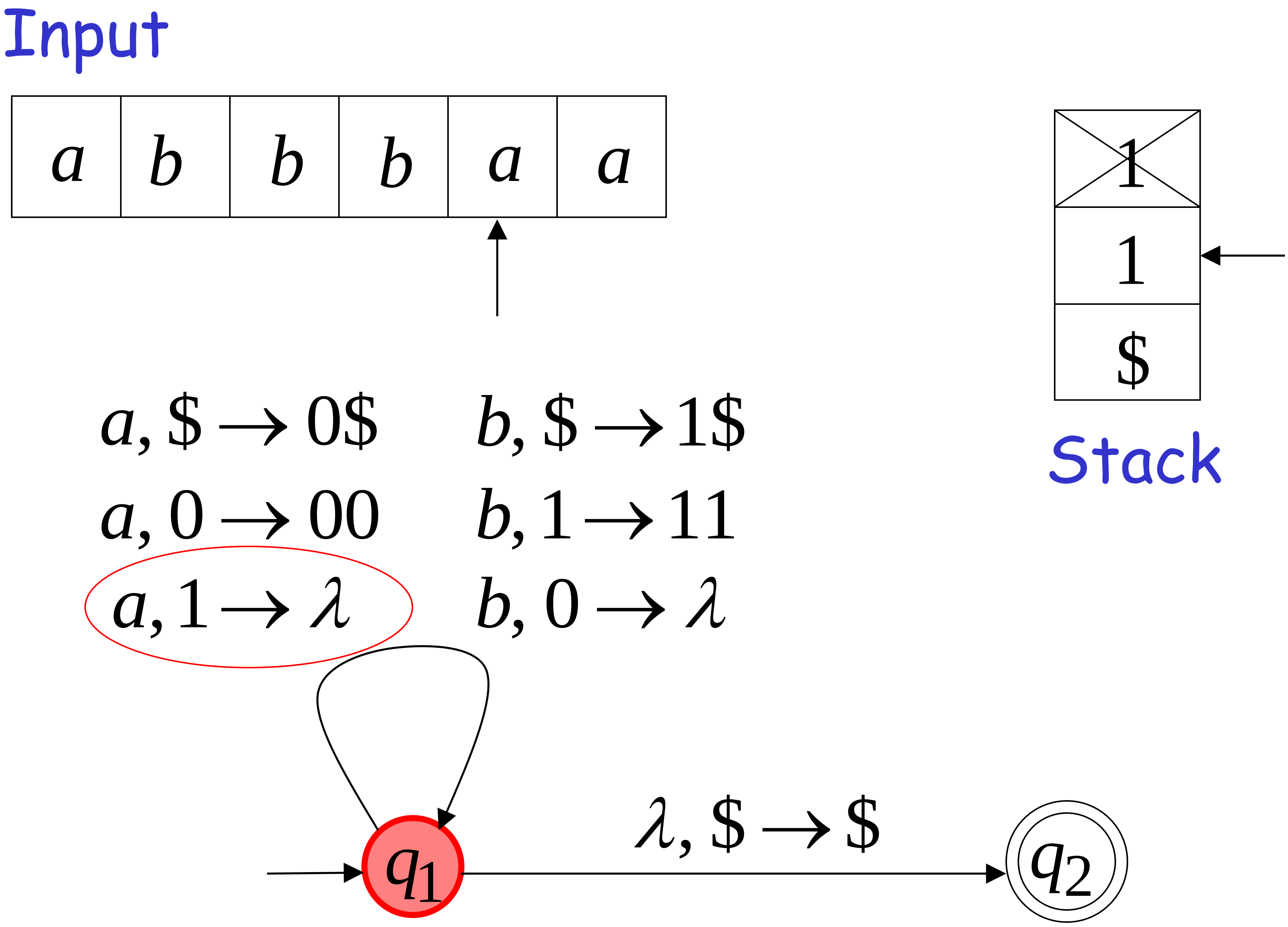
$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$

$$a, 1 \rightarrow \lambda \qquad b, 0 \rightarrow \lambda$$



Time 6


Input
a b b b a a
1
1
$
Stack
$a, \$ \rightarrow 0\$$
$b, \$ \rightarrow 1\$$
$a, 0 \rightarrow 00$
$b, 1 \rightarrow 11$
$a, 1 \rightarrow \lambda$
$b, 0 \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_1
q_2

Time 7

Input

a	b	b	b	a	a

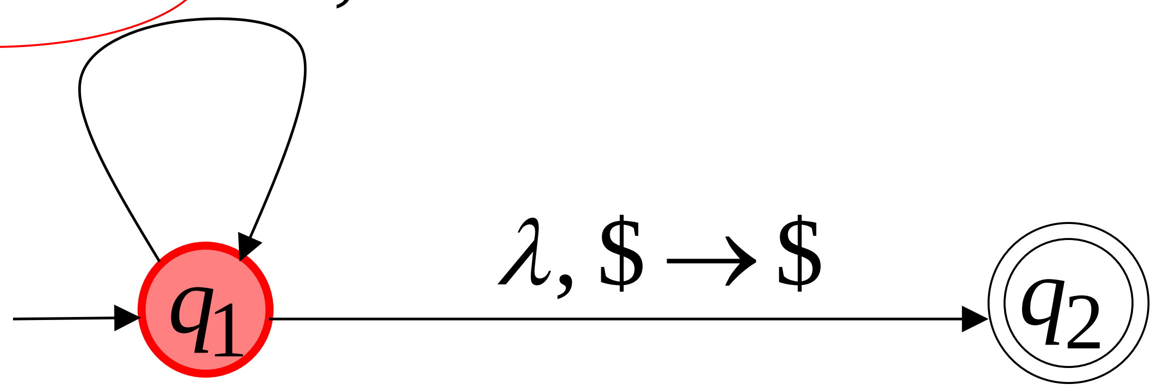
$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$

$$a, 1 \rightarrow \lambda \qquad b, 0 \rightarrow \lambda$$

$\cancel{1}$
$\$$

Stack



Time 8

Input

a	b	b	b	a	a

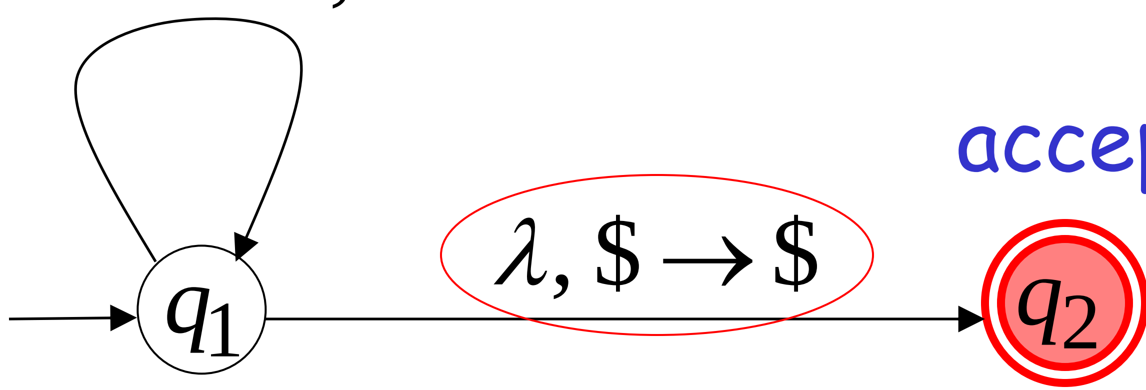
$\$$

Stack

$a, \$ \rightarrow 0\$$ $\quad$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $\quad$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda$ $\quad$ $b, 0 \rightarrow \lambda$

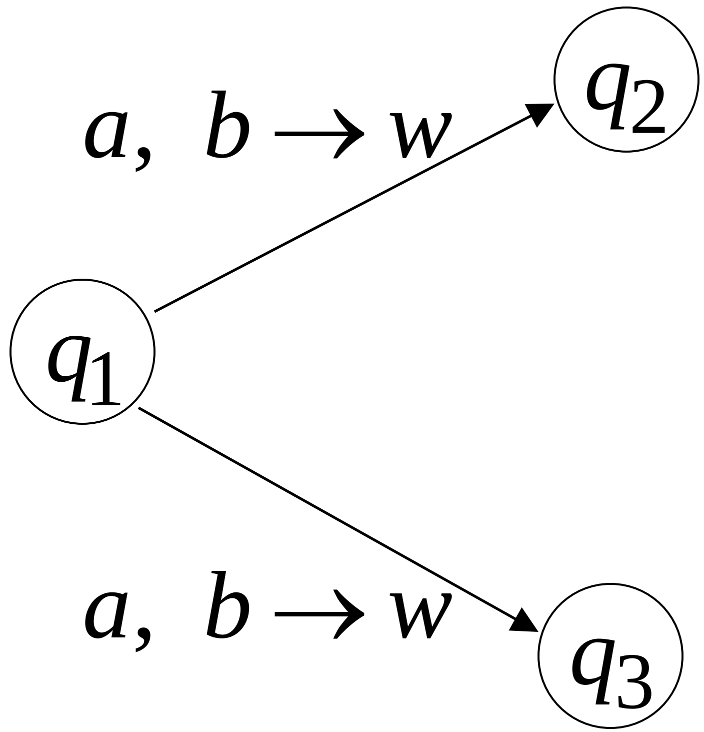


Formalities for NPDAs

$$q_1 \xrightarrow{a,\ b \to w} q_2$$

Transition function:

$$\delta(q_1, a, b) = \{(q_2, w)\}$$



Transition function:

$$\delta(q_1, a, b) = \{(q_2, w), \ (q_3, w)\}$$

Formal Definition

Non-Deterministic Pushdown Automaton
NPDA

$$M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$$

States

Input alphabet

Stack alphabet

Transition function

Initial state

Stack start symbol

Final states

Instantaneous Description

$$(q,\ u,\ s)$$

Current state

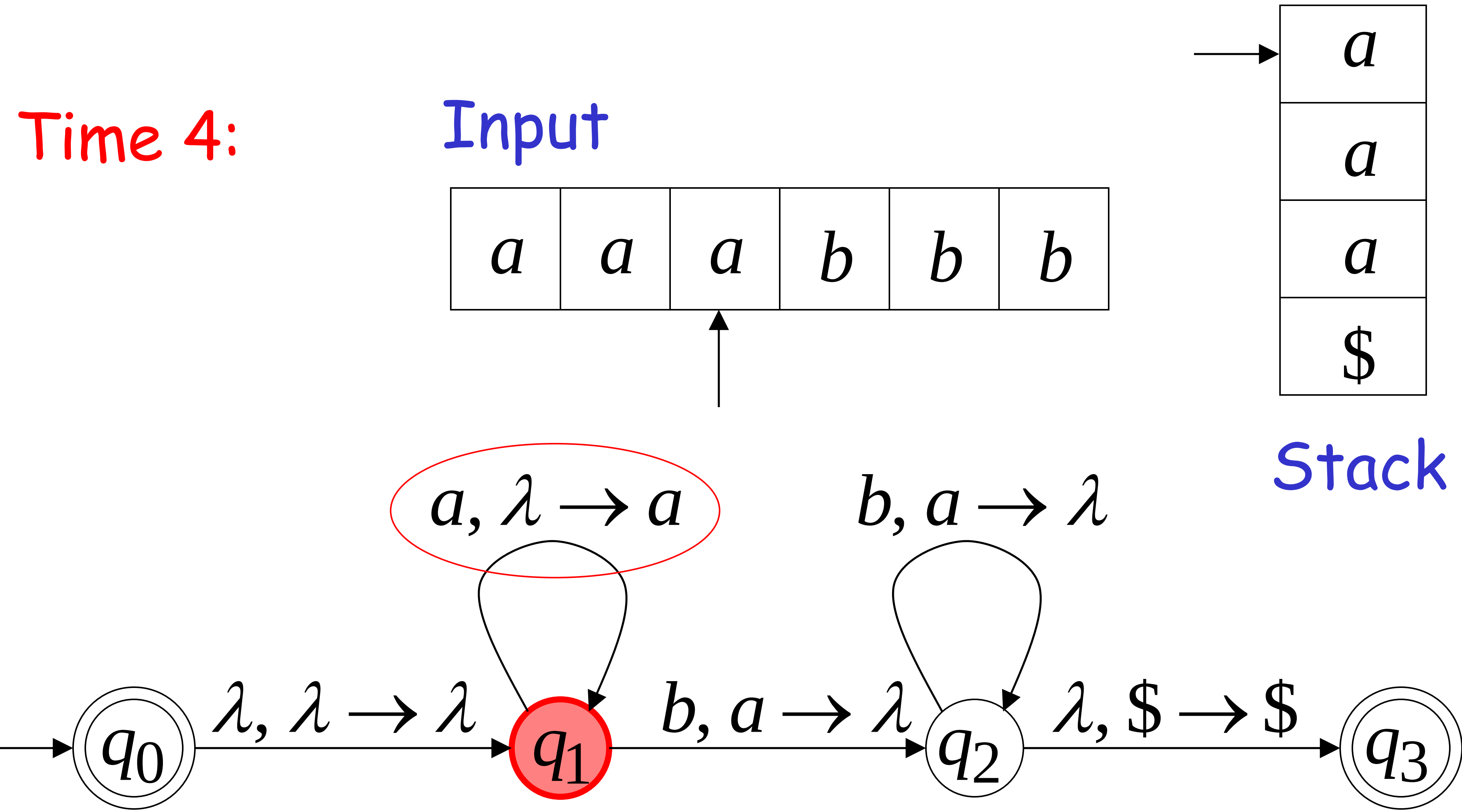
Remaining input

Current stack contents

Example: Instantaneous Description

$$(q_1, bbb, aaa\$)$$

Time 4:

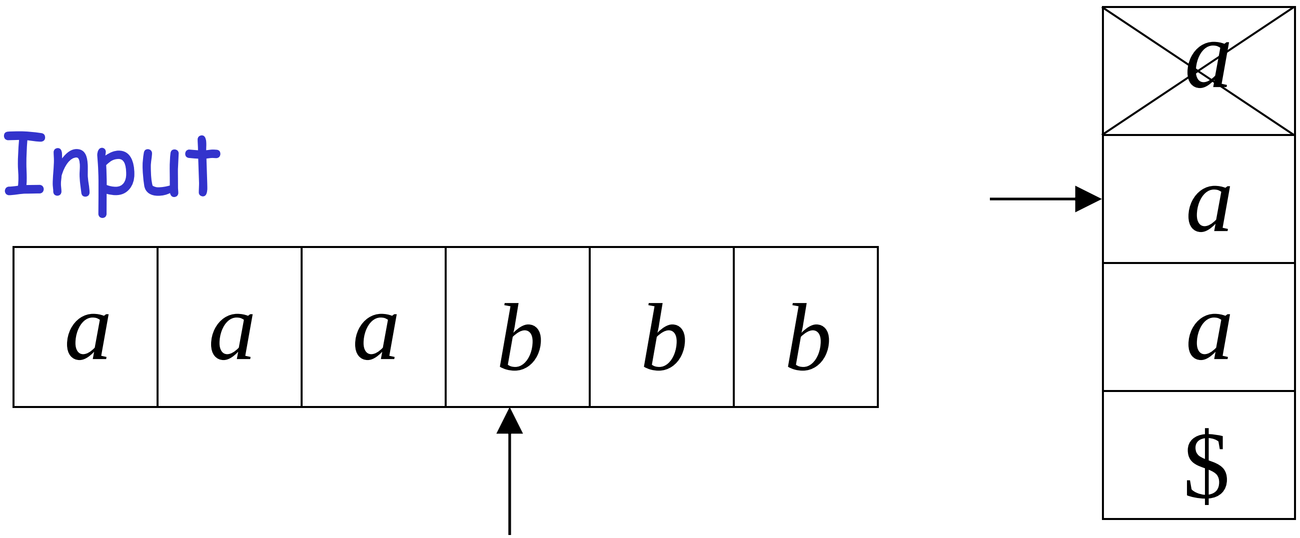


Example: **Instantaneous Description**

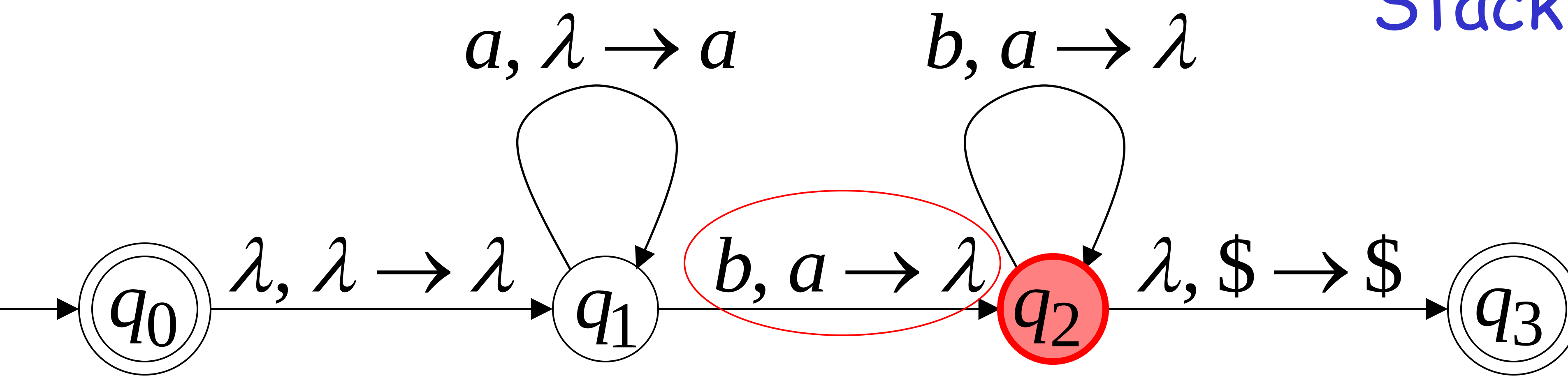
$$(q_2, bb, aa\$)$$

Time 5:

Input



Stack



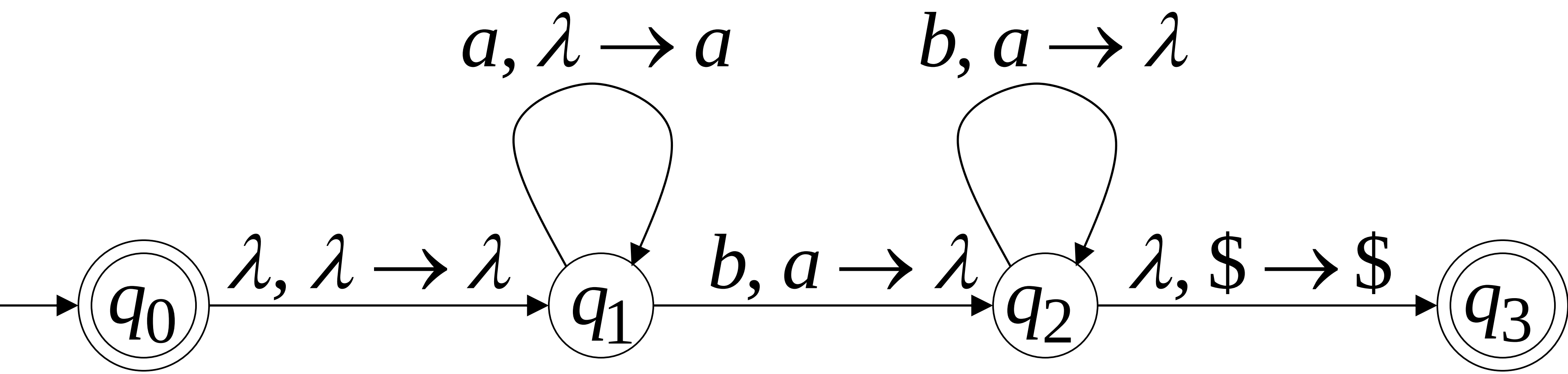
We write:

$$(q_1, bbb, aaa\$) \succ (q_2, bb, aa\$)$$

Time 4 Time 5

A computation:

$$(q_0, aaabbb, \$) \succ (q_1, aaabbb, \$) \succ$$
$$(q_1, aabbb, a\$) \succ (q_1, abbb, aa\$) \succ (q_1, bbb, aaa\$) \succ$$
$$(q_2, bb, aa\$) \succ (q_2, b, a\$) \succ (q_2, \lambda, \$) \succ (q_3, \lambda, \$)$$



$$(q_0, aaabbb, \$) \succ (q_1, aaabbb, \$) \succ$$
$$(q_1, aabbb, a\$) \succ (q_1, abbb, aa\$) \succ (q_1, bbb, aaa\$) \succ$$
$$(q_2, bb, aa\$) \succ (q_2, b, a\$) \succ (q_2, \lambda, \$) \succ (q_3, \lambda, \$)$$

For convenience we write:

$$(q_0, aaabbb, \$) \overset{*}{\succ} (q_3, \lambda, \$)$$

Formal Definition

Language $L(M)$ of NPDA M:

$$L(M) = \{w: \ (q_0, w, s) \overset{*}{\succ} (q_f, \lambda, s')\}$$

Initial state

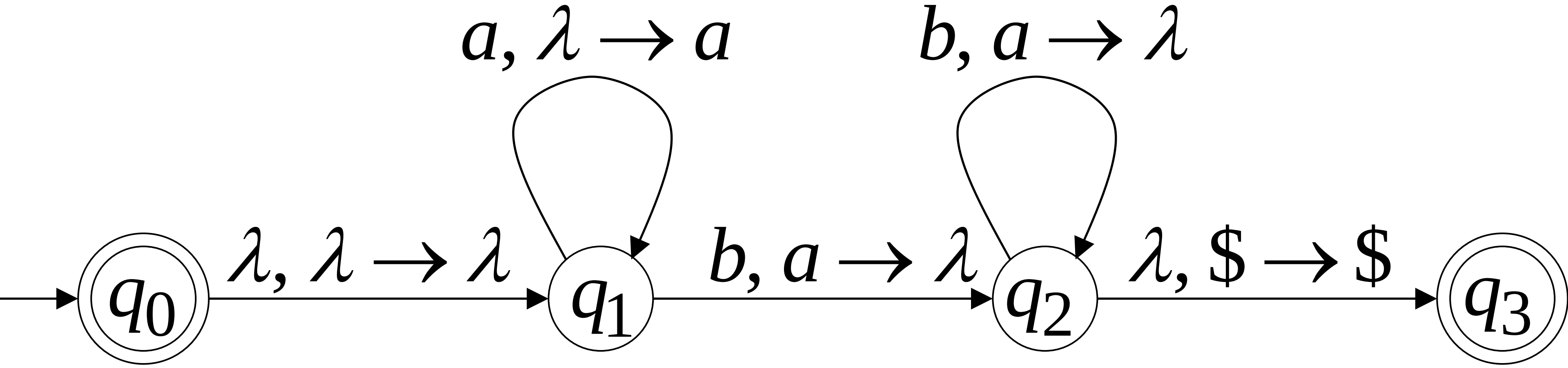
Final state

Example:

$$(q_0, aaabbb, \$) \overset{*}{\succ} (q_3, \lambda, \$)$$

$$aaabbb \in L(M)$$

NPDA M :

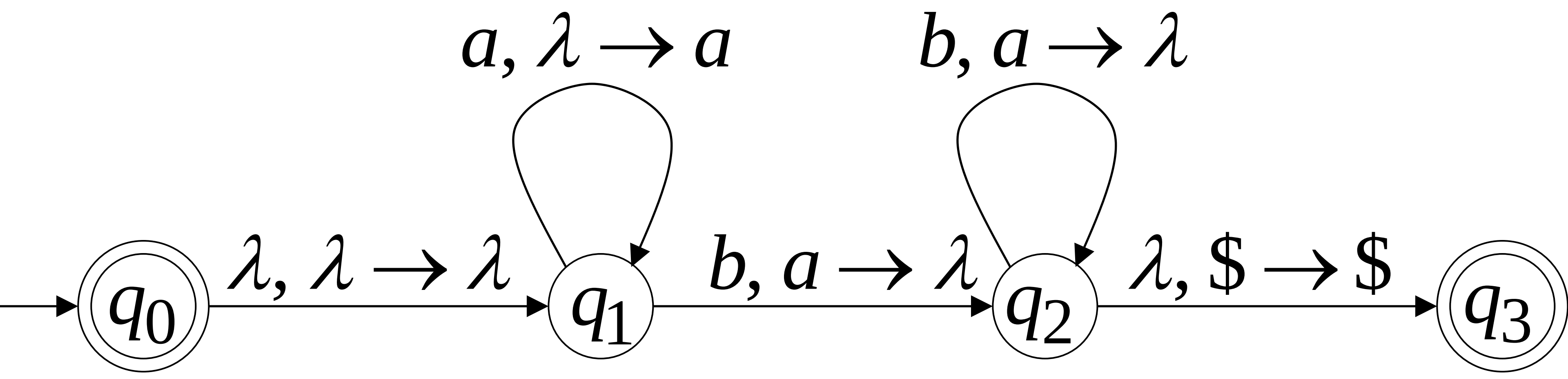


$$(q_0, a^n b^n, \$) \overset{*}{\succ} (q_3, \lambda, \$)$$

$$\Downarrow$$

$$a^n b^n \in L(M)$$

NPDA M :



Therefore: $L(M) = \{a^n b^n : n \geq 0\}$

NPDA M :