Single Final State for NFAs

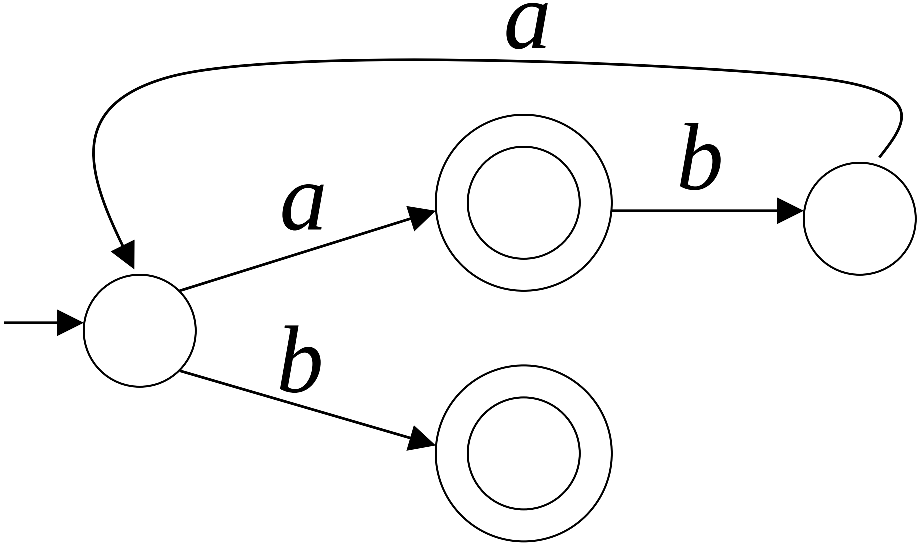
Any NFA can be converted

to an equivalent NFA

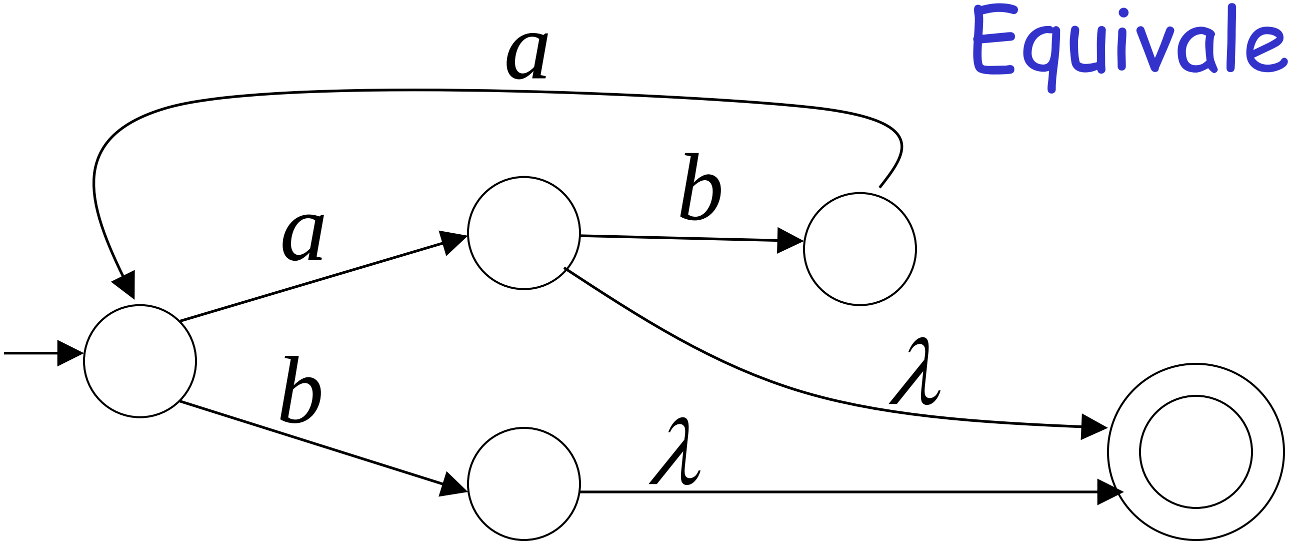
with a single final state

Example

NFA

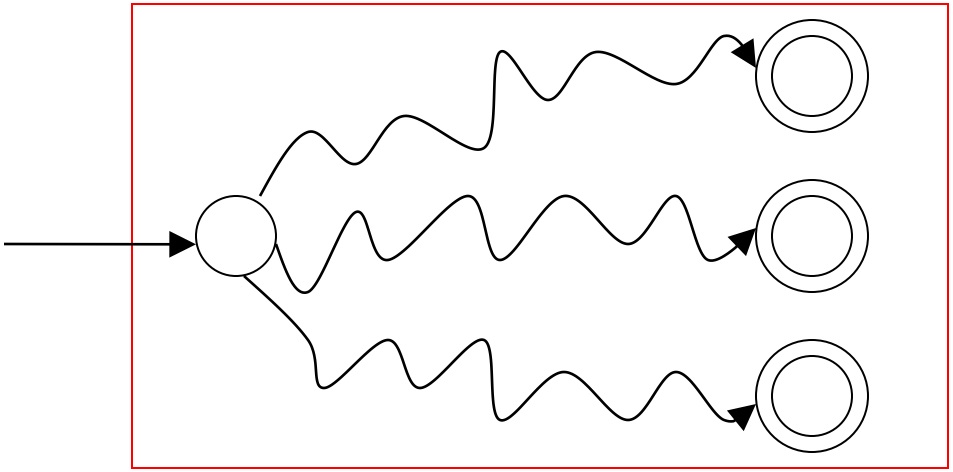


Equivalent NFA

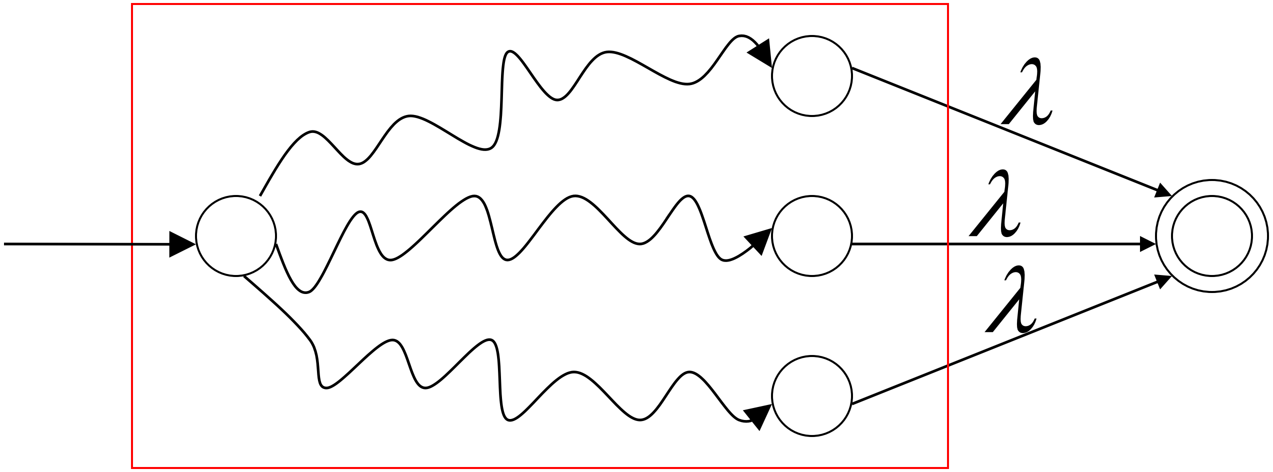


In General

NFA



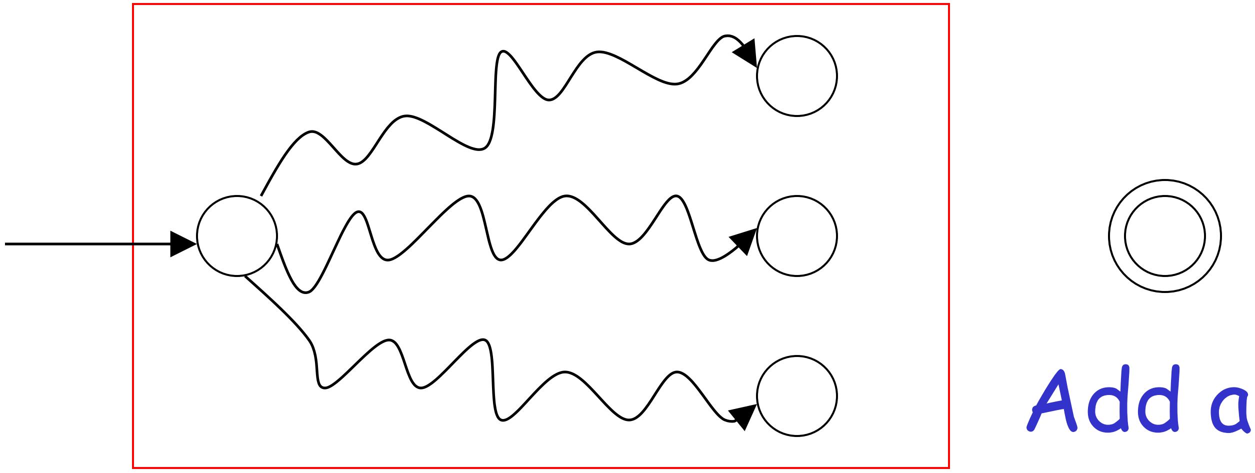
Equivalent NFA



Single
final state

Extreme Case

NFA without final state



Add a final state

Without transitions

Properties of Regular Languages

For regular languages L_1 and L_2 we will prove that:

$$\left.\begin{array}{rl}
\text{Union:} & L_1 \cup L_2 \\
\text{Concatenation:} & L_1 L_2 \\
\text{Star:} & L_1{}^* \\
\text{Reversal:} & L_1{}^R \\
\text{Complement:} & \overline{L_1} \\
\text{Intersection:} & L_1 \cap L_2
\end{array}\right\}$$

Are regular Languages

We say: Regular languages are **closed under**

Union: $L_1 \cup L_2$

Concatenation: $L_1 L_2$

Star: $L_1 *$

Reversal: L_1^R

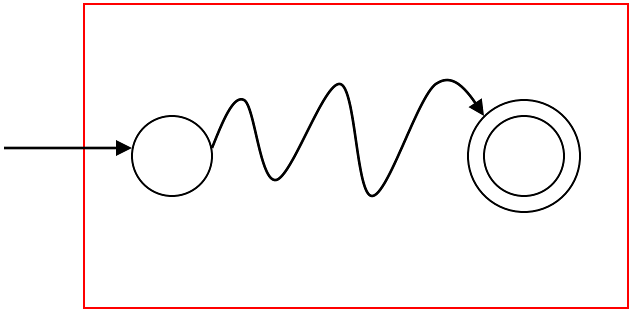
Complement: $\overline{L_1}$

Intersection: $L_1 \cap L_2$

Regular language L_1

$$L(M_1) = L_1$$

NFA M_1

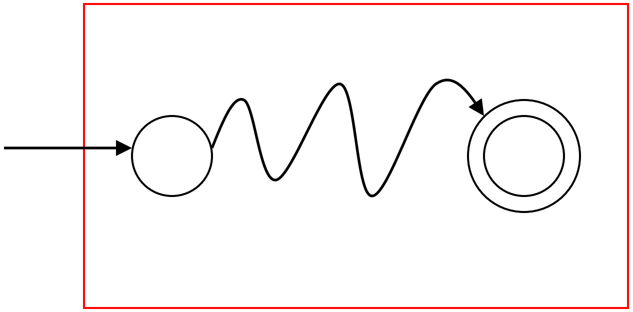


Single final state

Regular language L_2

$$L(M_2) = L_2$$

NFA M_2

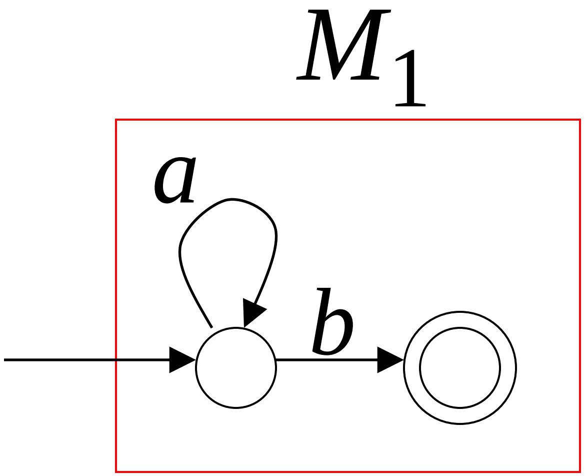


Single final state

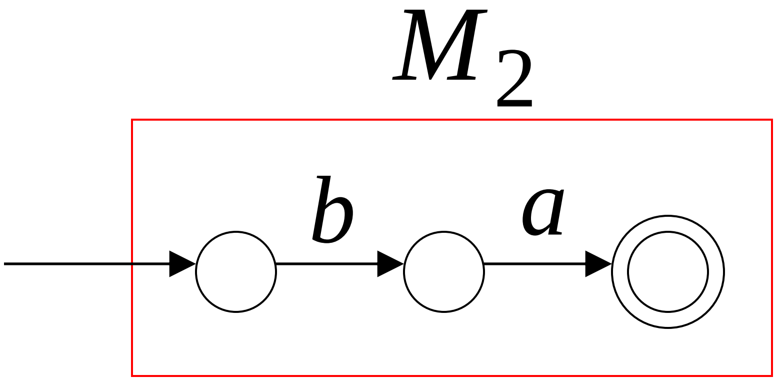
Example

$n \geq 0$

$$L_1 = \{a^n b\}$$

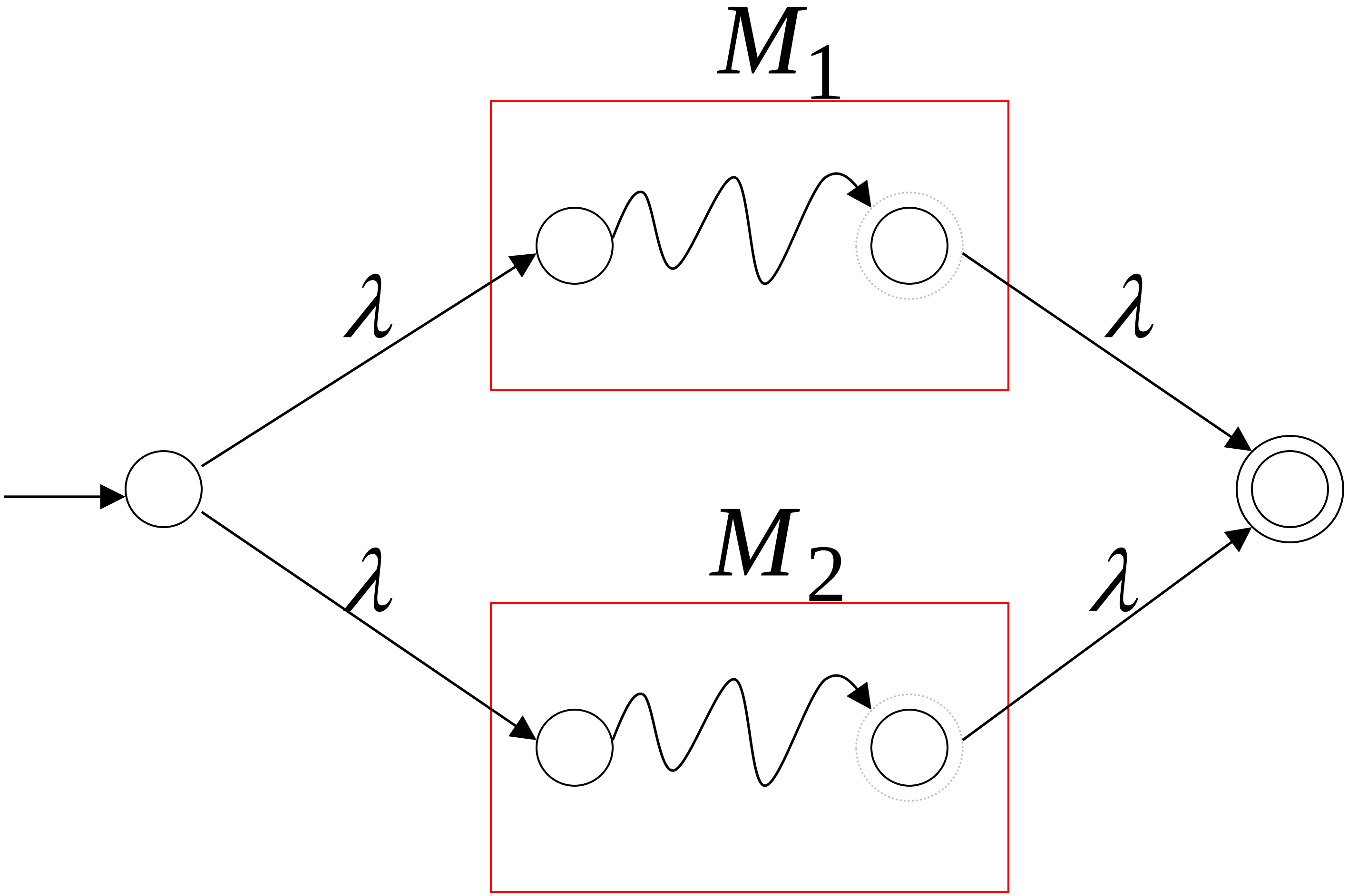


$$L_2 = \{ba\}$$



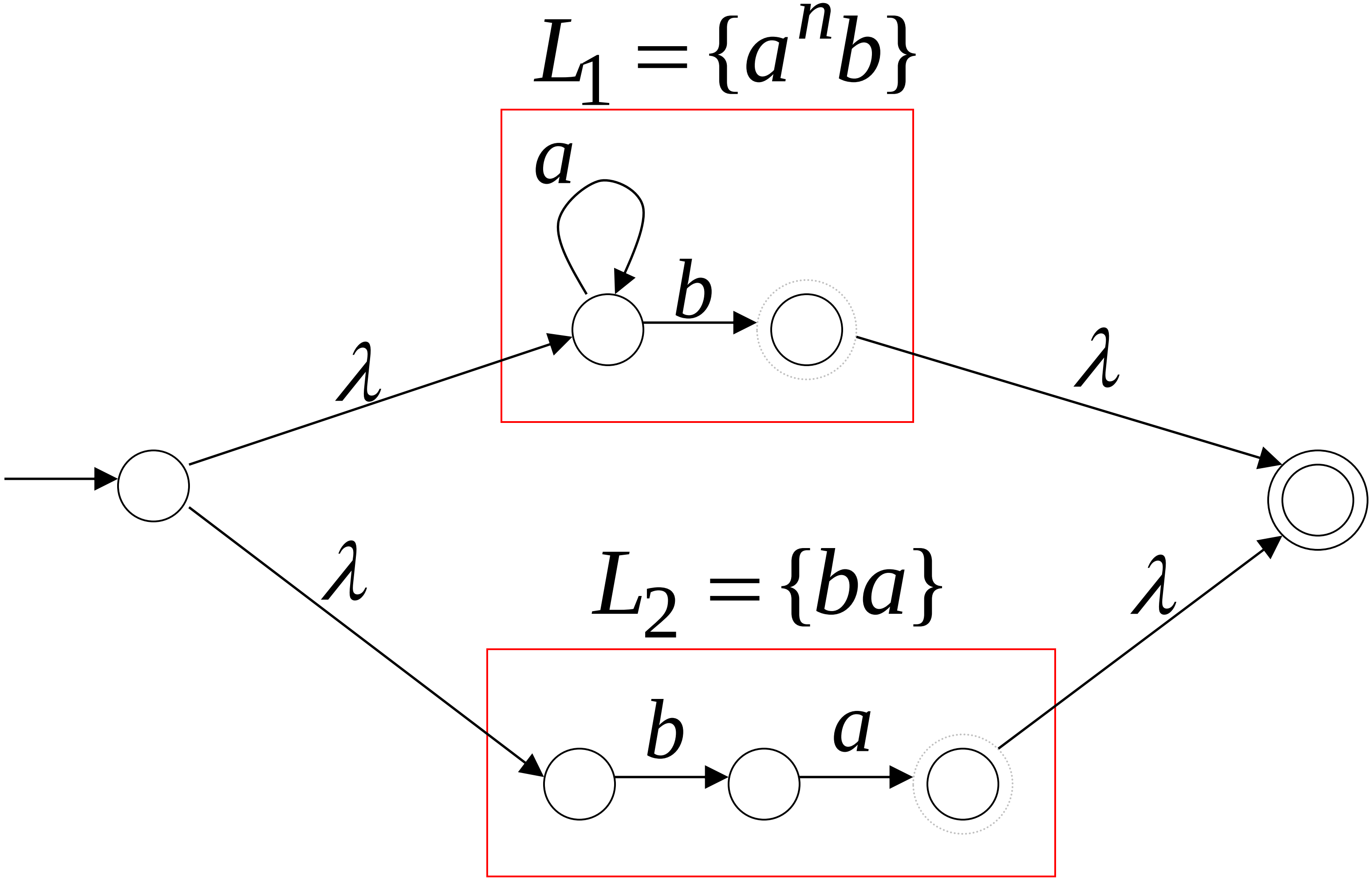
Union

NFA for $L_1 \cup L_2$



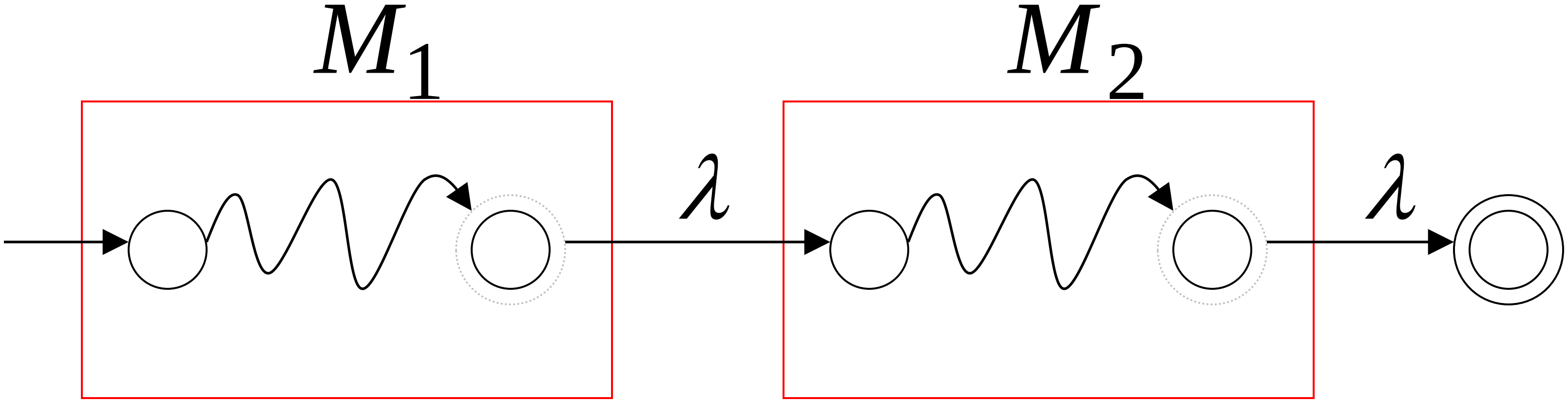
Example

NFA for $L_1 \cup L_2 = \{a^n b\} \cup \{ba\}$



Concatenation

NFA for $L_1 L_2$

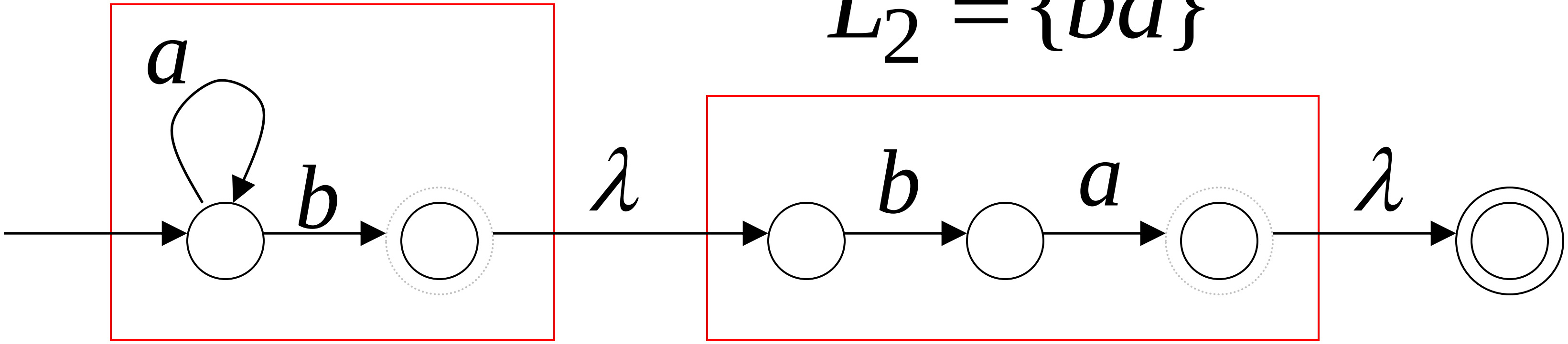


Example

NFA for $L_1 L_2 = \{a^n b\}\{ba\} = \{a^n bba\}$

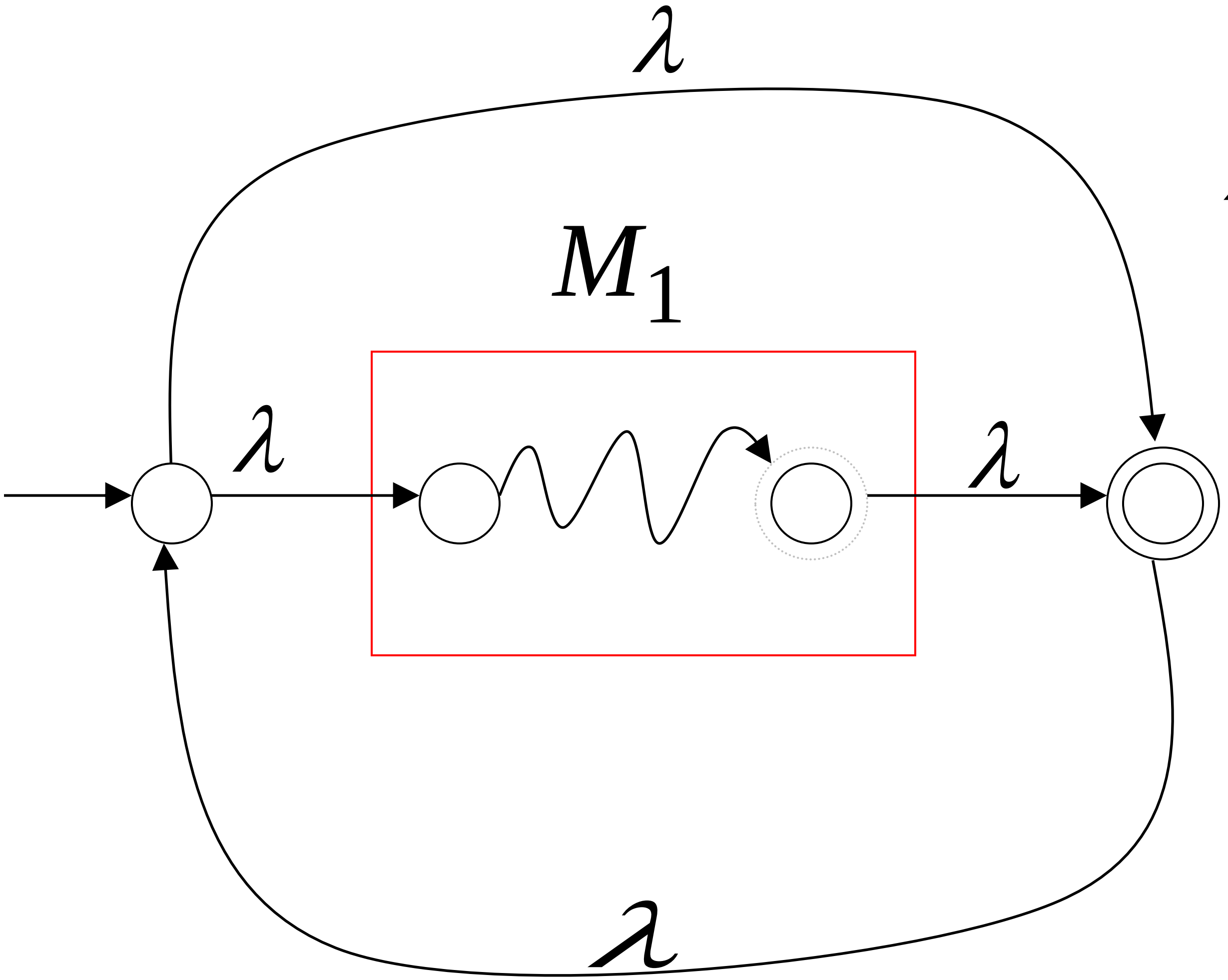
$L_1 = \{a^n b\}$

$L_2 = \{ba\}$



Star Operation

NFA for $L_1{}^*$



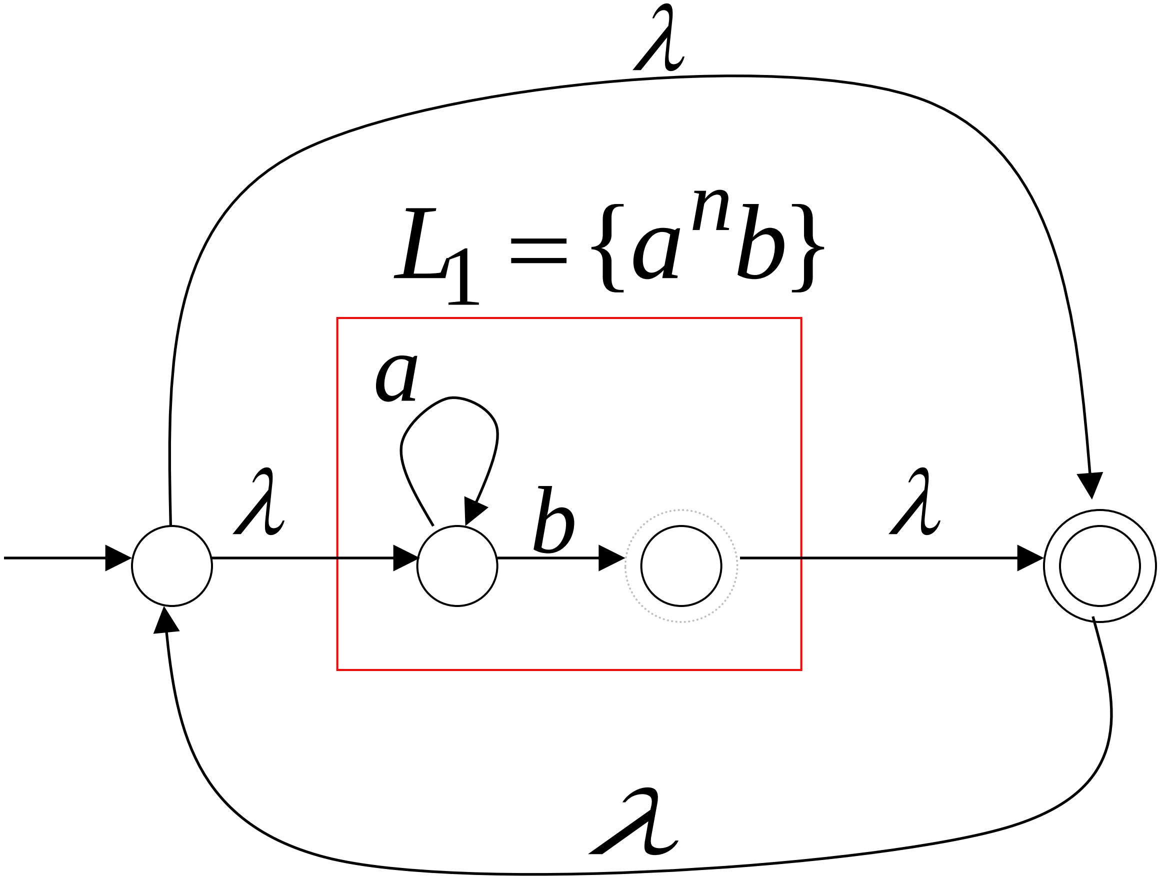
$\lambda \in L_1{}^*$

Example

NFA for $L_1{}^* = \{a^n b\}^*$

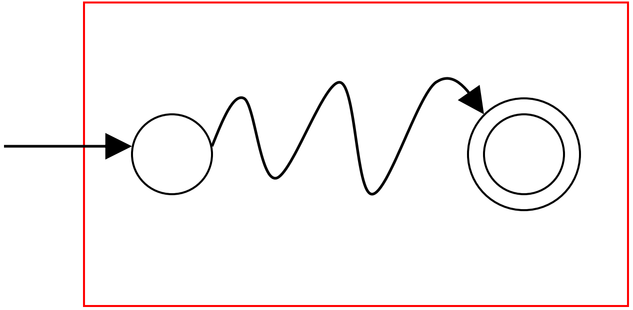
$w = w_1 w_2 \cdots w_k$

$w_i \in L_1$



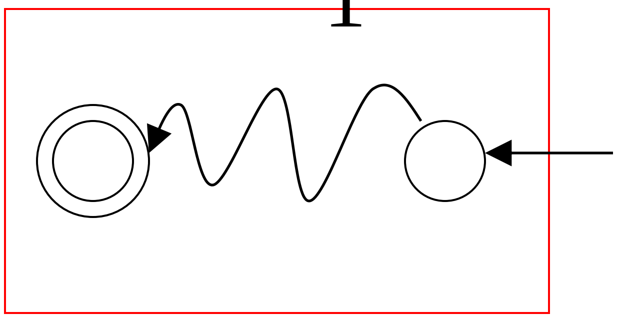
Reverse

L_1 M_1



NFA for L_1^R

M_1'



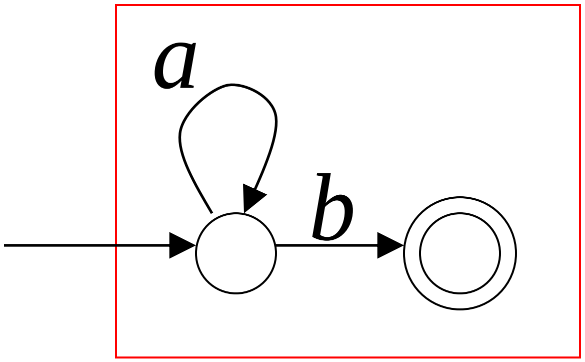
1. Reverse all transitions

2. Make initial state final state
and vice versa

Example

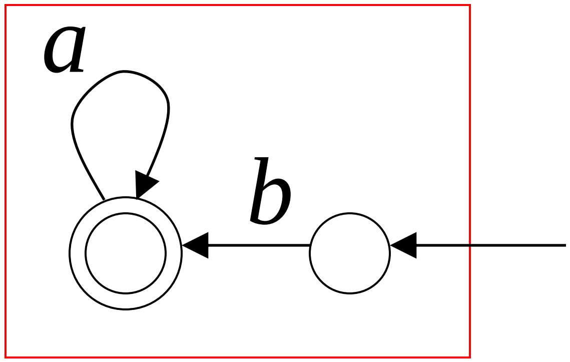
$$L_1 = \{a^n b\}$$

M_1

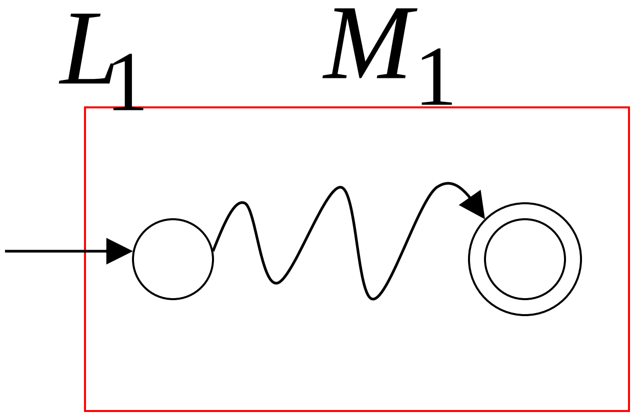


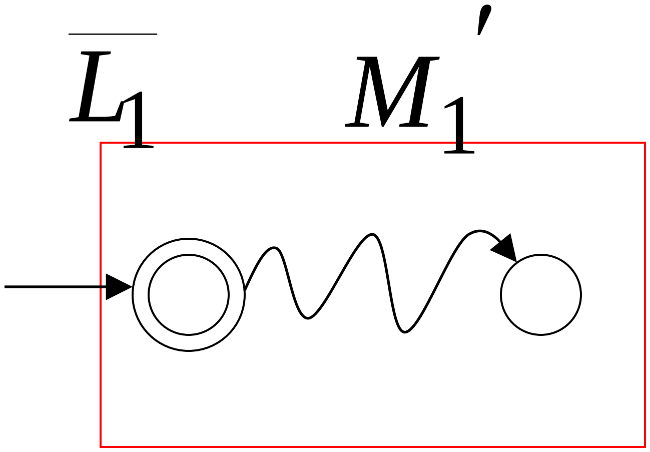
$$L_1^R = \{ba^n\}$$

M_1'



Complement



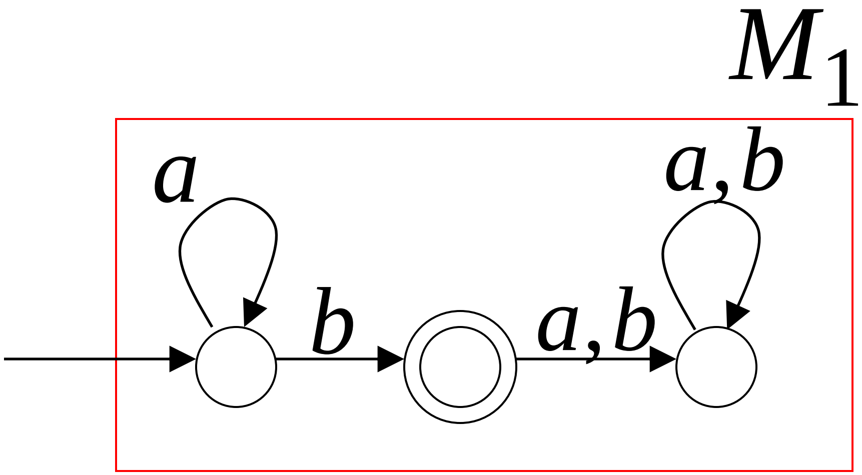


1. Take the **DFA** that accepts L_1

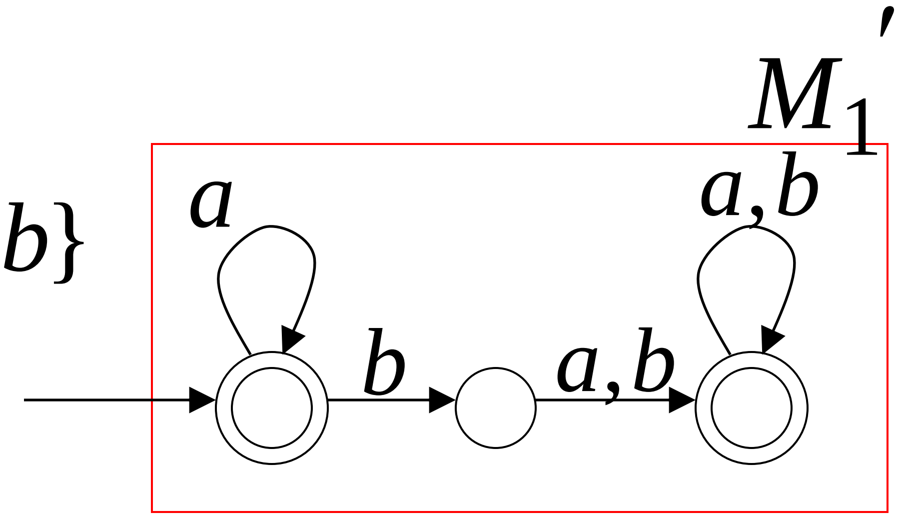
2. Make final states non-final,
and vice-versa

Example

$$L_1 = \{a^n b\}$$



$$\overline{L_1} = \{a,b\}^* - \{a^n b\}$$



Intersection

DeMorgan's Law: $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$

$L_1 \, , \; L_2$ regular

⇒ $\overline{L_1} \, , \; \overline{L_2}$ regular

⇒ $\overline{L_1} \cup \overline{L_2}$ regular

⇒ $\overline{\overline{L_1} \cup \overline{L_2}}$ regular

⇒ $L_1 \cap L_2$ regular

Example

$$\left.\begin{array}{ll} L_1 = \{a^n b\} & \text{regular} \\ L_2 = \{ab, ba\} & \text{regular} \end{array}\right\} \Rightarrow \begin{array}{c} L_1 \cap L_2 = \{ab\} \\ \text{regular} \end{array}$$

Regular Expressions

Regular Expressions

Regular expressions
describe regular languages

Example: $(a+b\cdot c)^*$

describes the language

$$\{a,bc\}^* = \{\lambda, a, bc, aa, abc, bca, \ldots\}$$

Recursive Definition

Primitive regular expressions: $\emptyset, \quad \lambda, \quad \alpha$

Given regular expressions r_1 and r_2

$$\left.\begin{array}{l} r_1 + r_2 \\ r_1 \cdot r_2 \\ r_1 * \\ (r_1) \end{array}\right\} \text{Are regular expressions}$$

Examples

A regular expression: $(a+b\cdot c)^*\cdot(c+\varnothing)$

Not a regular expression: $(a+b+)$

Languages of Regular Expressions

$L(r)$: language of regular expression r

Example

$$L((a+b\cdot c)^*) = \{\lambda, a, bc, aa, abc, bca, \ldots\}$$

Definition

For primitive regular expressions:

$$L(\varnothing) = \varnothing$$

$$L(\lambda) = \{\lambda\}$$

$$L(a) = \{a\}$$

Definition (continued)

For regular expressions r_1 and r_2

$$L(r_1 + r_2) = L(r_1) \cup L(r_2)$$

$$L(r_1 \cdot r_2) = L(r_1)\, L(r_2)$$

$$L(r_1 *) = (L(r_1))*$$

$$L((r_1)) = L(r_1)$$

Example

Regular expression: $(a+b)\cdot a^*$

$$
\begin{aligned}
L((a+b)\cdot a^*) &= L((a+b))\,L(a^*) \\
&= L(a+b)\,L(a^*) \\
&= (L(a)\cup L(b))\,(L(a))^* \\
&= (\{a\}\cup\{b\})\,(\{a\})^* \\
&= \{a,b\}\,\{\lambda,a,aa,aaa,\ldots\} \\
&= \{a,aa,aaa,\ldots,b,ba,baa,\ldots\}
\end{aligned}
$$

Example

Regular expression $r = (a+b)^*(a+bb)$

$$L(r) = \{a, bb, aa, abb, ba, bbb, \ldots\}$$

Example

Regular expression $r = (aa)^*(bb)^*b$

$$L(r) = \{a^{2n}b^{2m}b: \quad n, m \geq 0\}$$

Example

Regular expression $r = (0+1)^* \, 00 \, (0+1)^*$

$L(r)$ = { all strings with at least two consecutive 0 }

Example

Regular expression $r = (1+01)^*(0+\lambda)$

$L(r)$ = { all strings without two consecutive 0 }

Equivalent Regular Expressions

Definition:

Regular expressions r_1 and r_2

are **equivalent** if $L(r_1) = L(r_2)$

Example

L = { all strings without
two consecutive 0 }

$$r_1 = (1+01)^*(0+\lambda)$$

$$r_2 = (1^*011^*)^*(0+\lambda)+1^*(0+\lambda)$$

$L(r_1) = L(r_2) = L$ ⇒ r_1 and r_2 are equivalent regular expr.

Regular Expressions
and
Regular Languages

Theorem

$$\left\{\begin{array}{l}\text{Languages}\\ \text{Generated by}\\ \text{Regular Expressions}\end{array}\right\} = \left\{\begin{array}{l}\text{Regular}\\ \text{Languages}\end{array}\right\}$$

Theorem - Part 1

$$\left\{\begin{array}{l}\text{Languages}\\ \text{Generated by}\\ \text{Regular Expressions}\end{array}\right\} \subseteq \left\{\begin{array}{l}\text{Regular}\\ \text{Languages}\end{array}\right\}$$

1. For any regular expression r
   the language $L(r)$ is regular

Theorem - Part 2

$$\left\{\begin{array}{l}\text{Languages}\\ \text{Generated by}\\ \text{Regular Expressions}\end{array}\right\} \supseteq \left\{\begin{array}{l}\text{Regular}\\ \text{Languages}\end{array}\right\}$$

2. For any regular language L there is a regular expression r with $L(r) = L$

Proof - Part 1

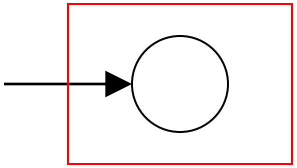
1. For any regular expression r
the language $L(r)$ is regular

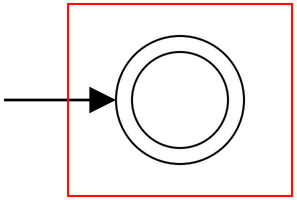
Proof by induction on the size of r

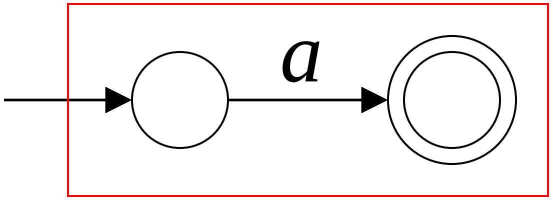
Induction Basis

Primitive Regular Expressions: $\emptyset, \quad \lambda, \quad \alpha$

NFAs







$$\left.\begin{array}{c} L(M_1) = \emptyset = L(\emptyset) \\[2em] L(M_2) = \{\lambda\} = L(\lambda) \\[2em] L(M_3) = \{a\} = L(a) \end{array}\right\} \text{regular languages}$$

Inductive Hypothesis

Assume

for regular expressions r_1 and r_2

that

$L(r_1)$ and $L(r_2)$ are regular languages

Inductive Step

We will prove:

$$L(r_1 + r_2)$$

$$L(r_1 \cdot r_2)$$

$$L(r_1 *)$$

$$L((r_1))$$

Are regular Languages

By definition of regular expressions:

$$L(r_1 + r_2) = L(r_1) \cup L(r_2)$$

$$L(r_1 \cdot r_2) = L(r_1)\, L(r_2)$$

$$L(r_1 *) = (L(r_1))*$$

$$L((r_1)) = L(r_1)$$

By inductive hypothesis we know:

$L(r_1)$ and $L(r_2)$ are regular languages

We also know:

Regular languages are closed under:

Union	$L(r_1) \cup L(r_2)$
Concatenation	$L(r_1)\,L(r_2)$
Star	$(L(r_1))^*$

Therefore:

$$\left.\begin{aligned} L(r_1 + r_2) &= L(r_1) \cup L(r_2) \\ L(r_1 \cdot r_2) &= L(r_1)\, L(r_2) \\ L(r_1 *) &= (L(r_1))* \end{aligned}\right\} \text{Are regular languages}$$

And trivially:

$L((r_1))$ is a regular language

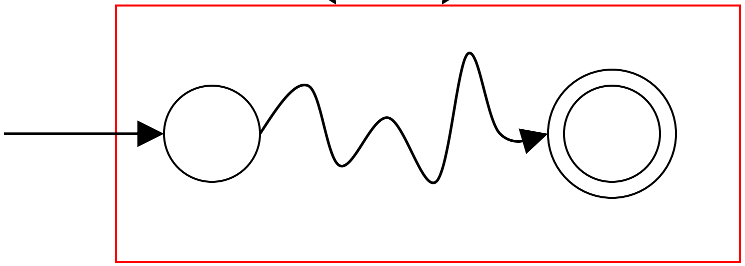
Proof - Part 2

2. For any regular language L there is a regular expression r with $L(r) = L$

Proof by construction of regular expression

Since L is regular take the NFA M that accepts it

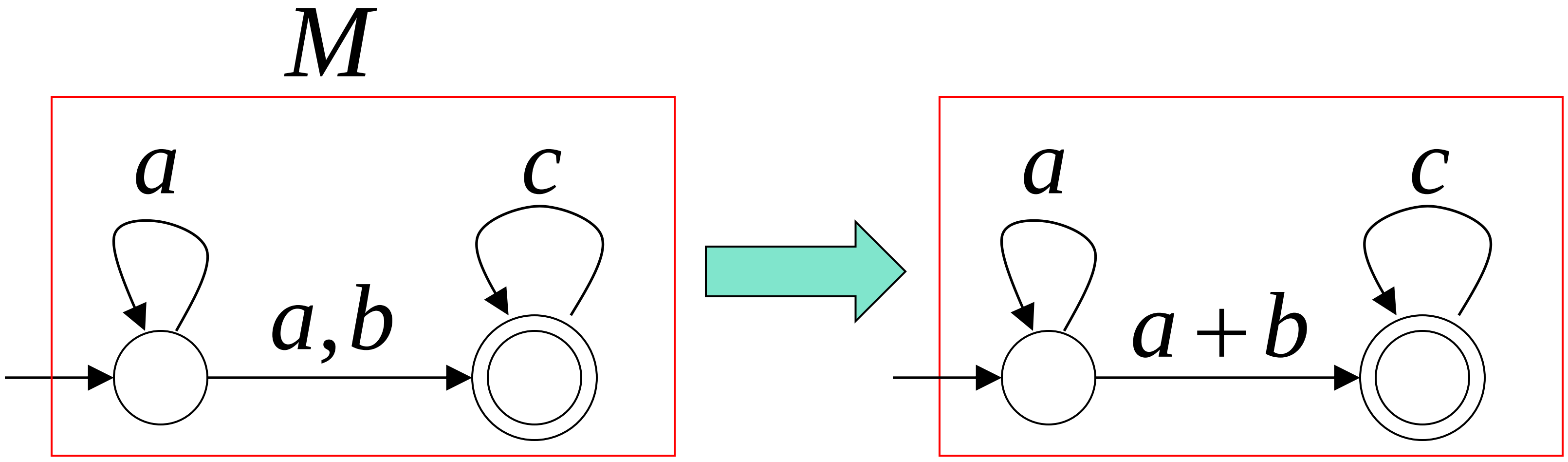
$$L(M) = L$$



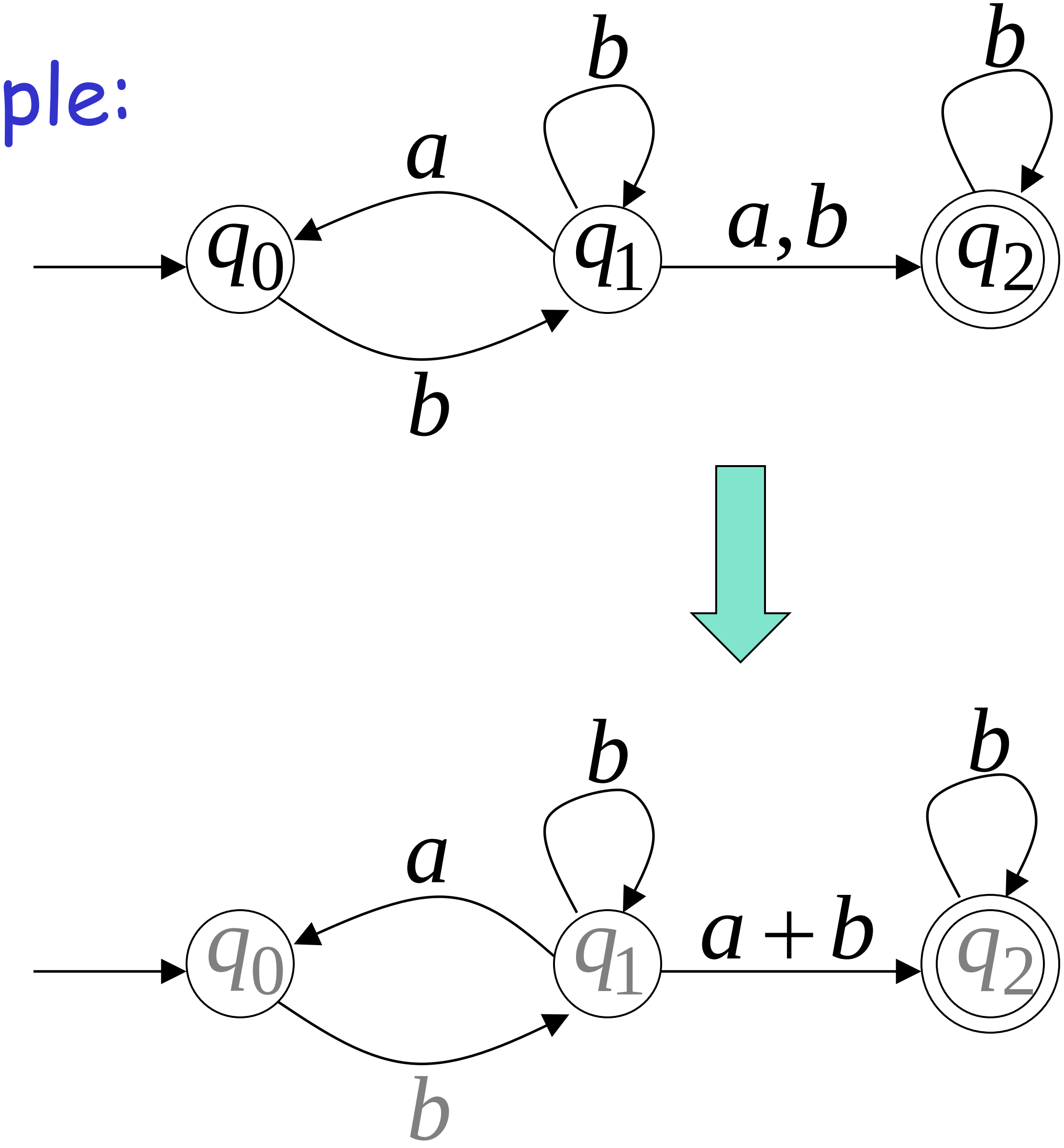
Single final state

From M construct the equivalent
Generalized Transition Graph
in which transition labels are regular expressions

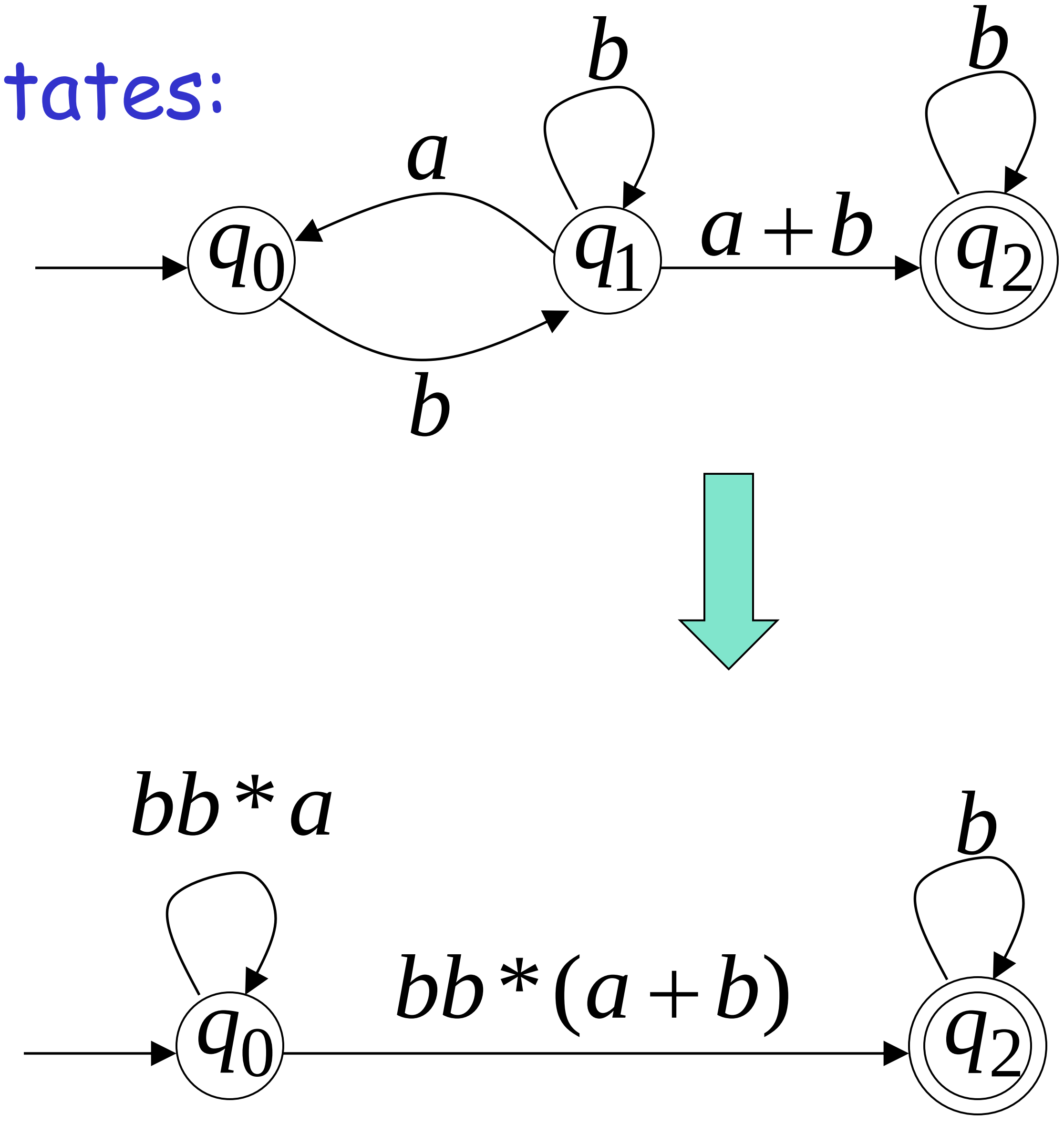
Example:



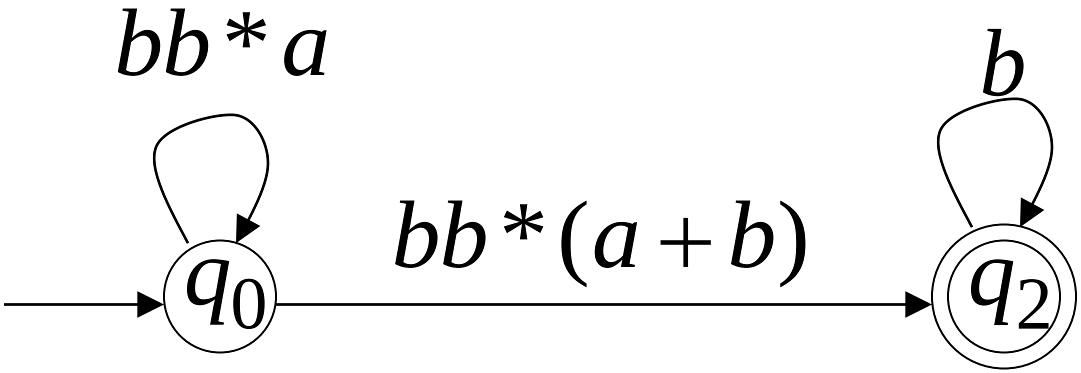
Another Example:



Reducing the states:



Resulting Regular Expression:

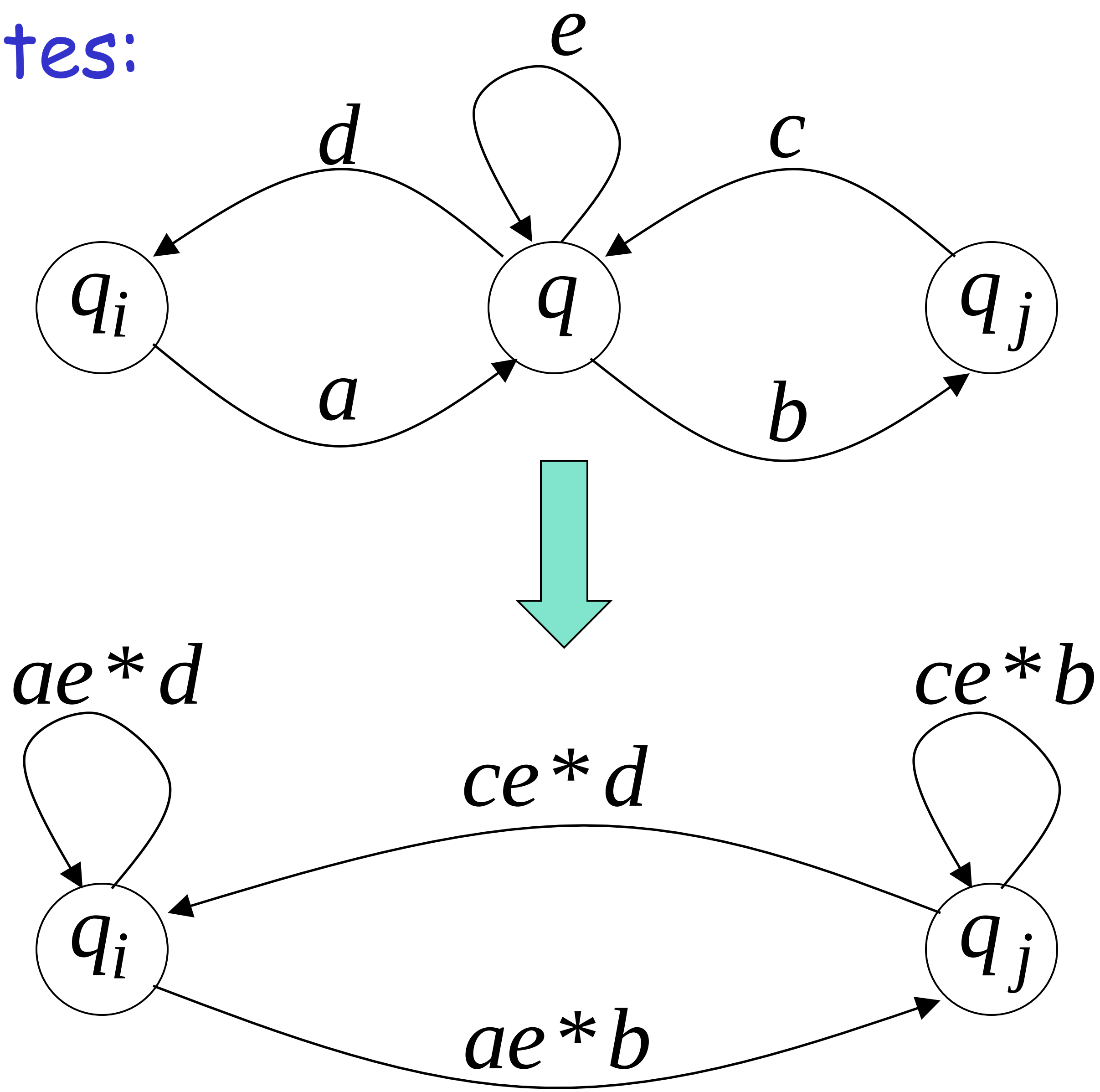


$$r = (bb^*a)^*bb^*(a+b)b^*$$

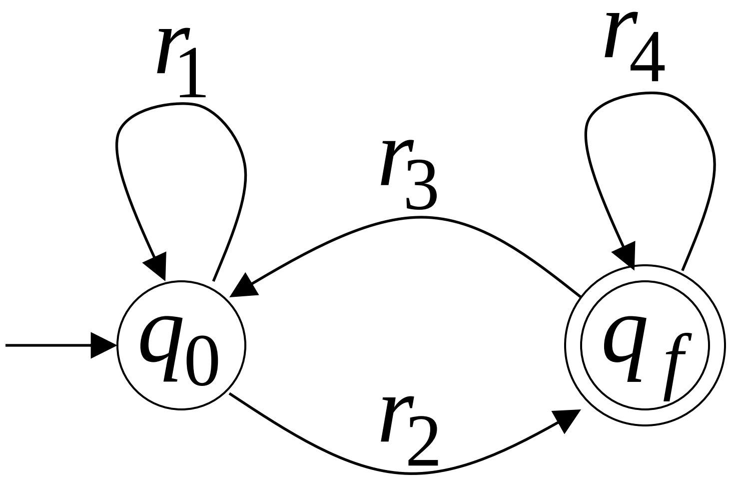
$$L(r) = L(M) = L$$

In General

Removing states:



The final transition graph:

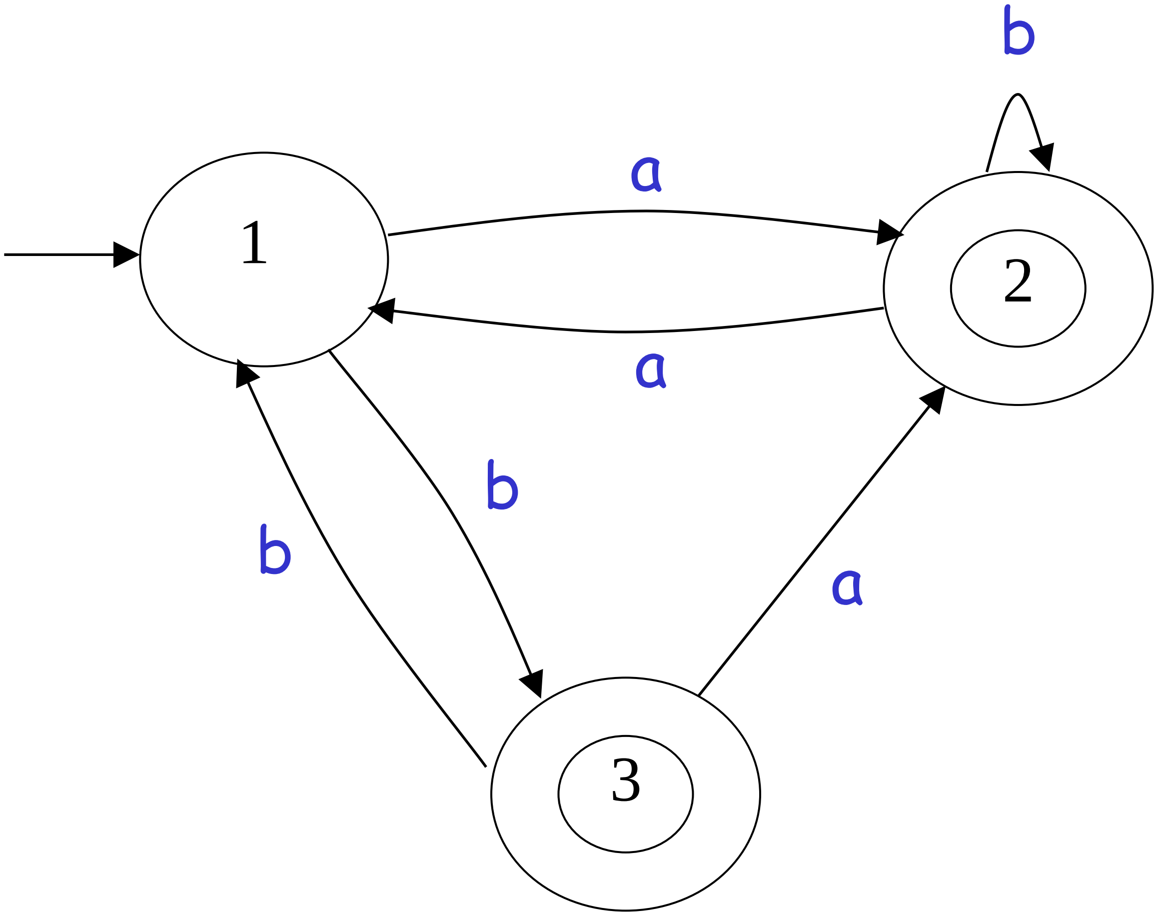


The resulting regular expression:

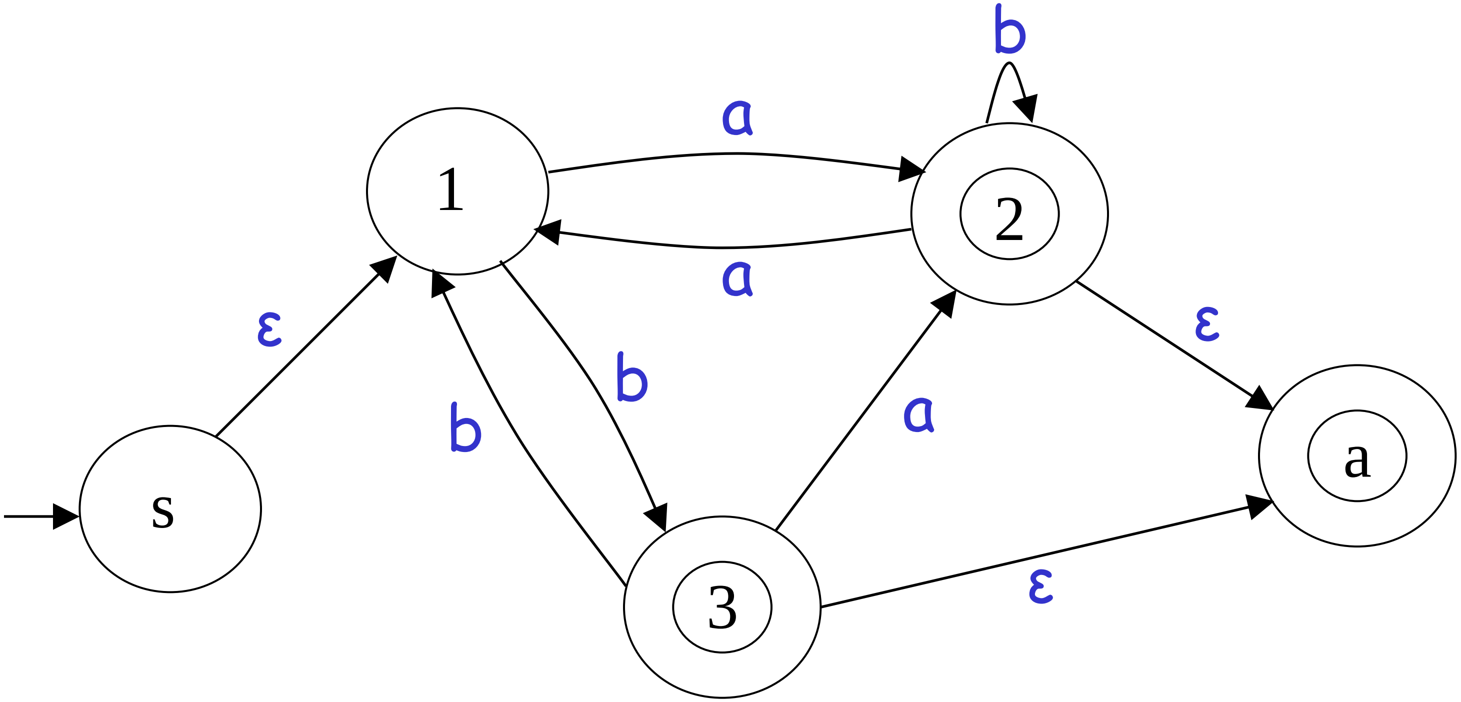
$$r = r_1 * r_2 (r_4 + r_3 r_1 * r_2) *$$

$$L(r) = L(M) = L$$

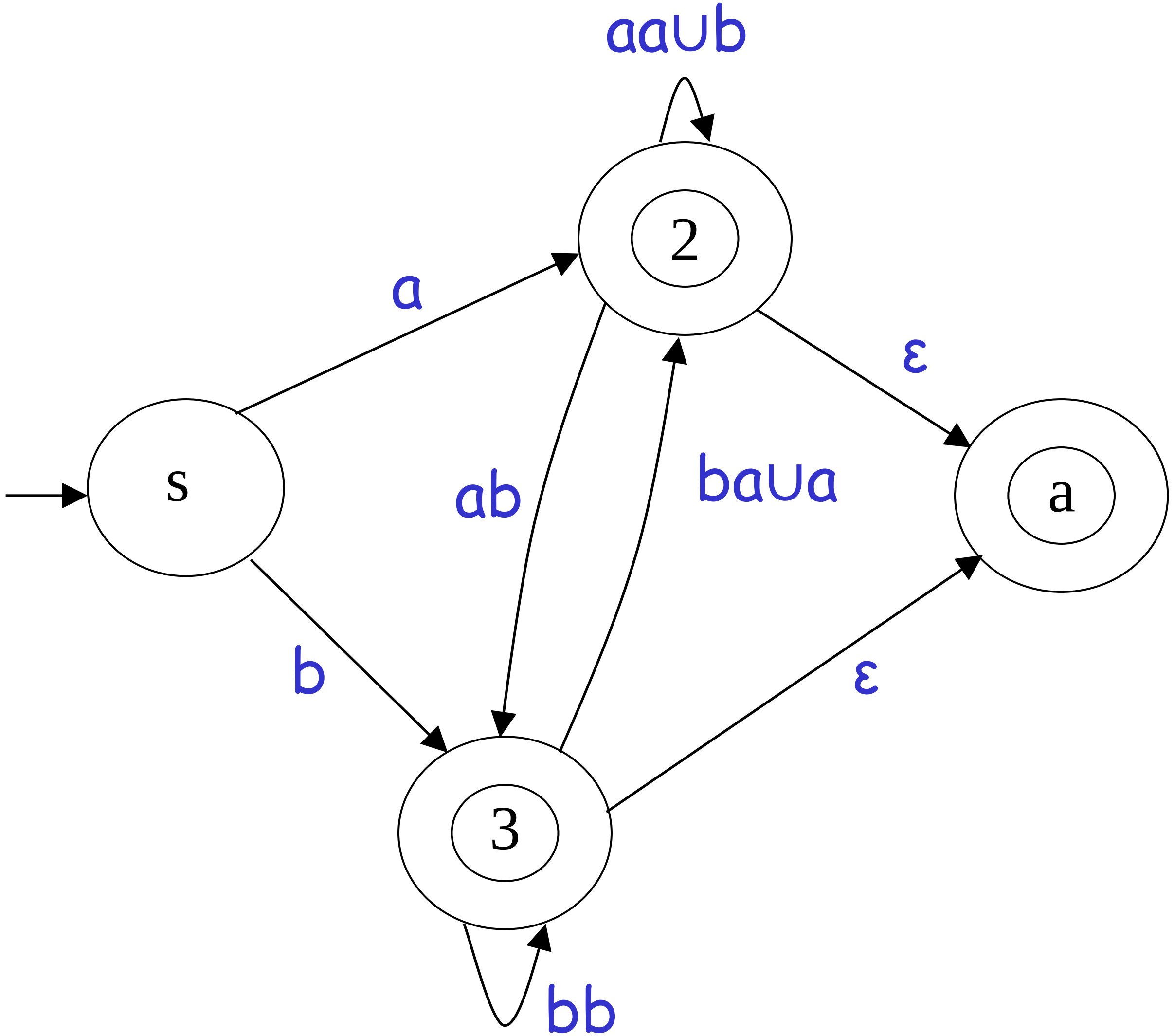
Example



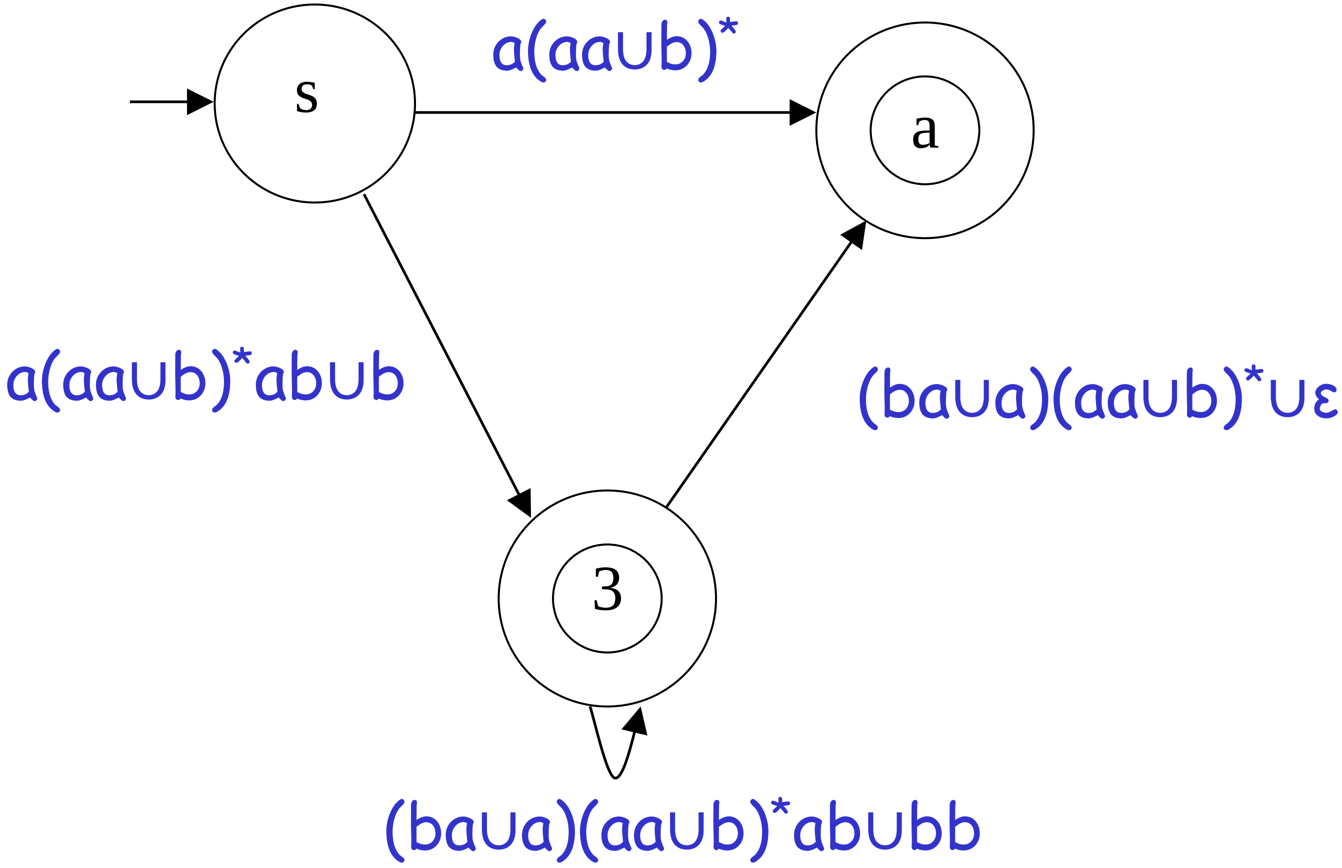
Example (Cont.)



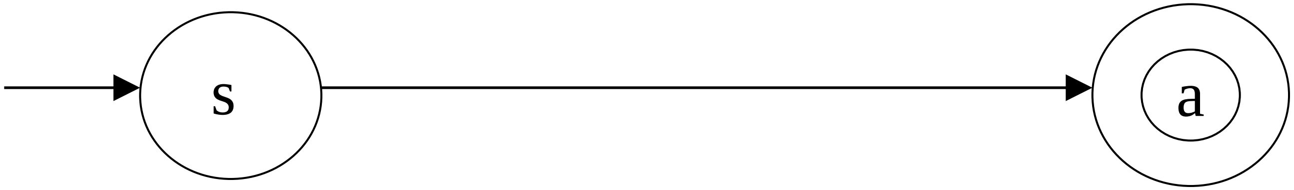
Example (Cont.)



Example (Cont.)



Example (Cont.)



$(a(aa \cup b)^*ab \cup b)((ba \cup a)(aa \cup b)^*ab \cup bb)^*((ba \cup a)(aa \cup b)^* \cup \varepsilon) \cup a(aa \cup b)^*$

The End