More Applications

of

the Pumping Lemma

The Pumping Lemma:

- Given a infinite regular language L

- there exists an integer m

- for any string $w \in L$ with length $|w| \geq m$

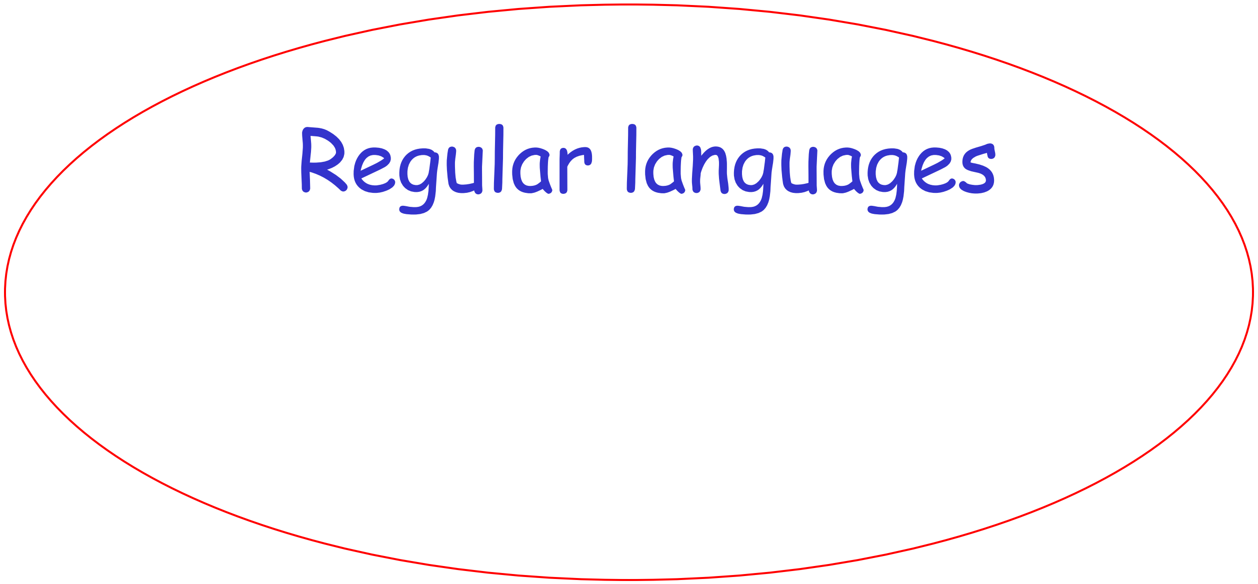
- we can write $w = x\,y\,z$

- with $|x\,y| \leq m$ and $|y| \geq 1$

- such that: $x\,y^i\,z \in L \qquad i = 0, 1, 2, \ldots$

Non-regular languages

$$L = \{vv^R : v \in \Sigma^*\}$$



Regular languages

Theorem: The language

$$L = \{vv^R : v \in \Sigma^*\} \qquad \Sigma = \{a, b\}$$

is not regular

Proof: Use the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Assume for contradiction that L is a regular language

Since L is infinite we can apply the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$ and

length $|w| \geq m$

We pick $w = a^m b^m b^m a^m$

Write $a^m b^m b^m a^m = x\,y\,z$

From the Pumping Lemma

it must be that length $|x\,y| \leq m, \quad |y| \geq 1$

$$xyz = \overbrace{a...aa...a...a}^{m}\overbrace{b...b}^{m}\overbrace{b...b}^{m}\overbrace{a...a}^{m}$$

$$xyz = \underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{...ab...bb...ba...a}_{z}$$

Thus: $y = a^k, \quad k \geq 1$

$x\,y\,z = a^m b^m b^m a^m$ $\qquad$ $y = a^k, \quad k \geq 1$

From the Pumping Lemma: $\quad x\,y^i\,z \;\in\; L$

$$i = 0, 1, 2, \ldots$$

Thus: $\quad x\,y^2\,z \;\in\; L$

$x\,y\,z = a^m b^m b^m a^m \qquad\qquad y = a^k, \quad k \geq 1$

From the Pumping Lemma: $x\,y^2\,z \in L$

$$xy^2z = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}a...}^{m+k}\underbrace{a\overbrace{b...b}^{m}\overbrace{b...b}^{m}\overbrace{a...a}^{m}}_{z} \in L$$

Thus: $a^{m+k} b^m b^m a^m \in L$

$$a^{m+k} b^m b^m a^m \quad \in L \qquad\qquad k \geq 1$$

BUT: $L = \{vv^R : v \in \Sigma^*\}$

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$$a^{m+k} b^m b^m a^m \quad \notin L$$

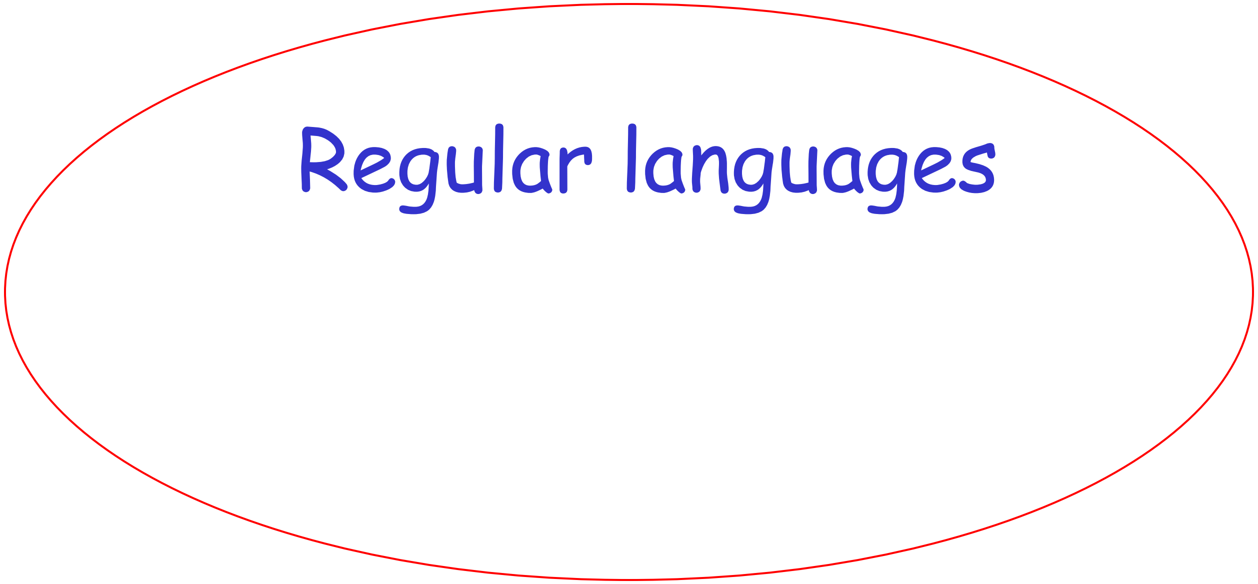
CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

Non-regular languages

$$L = \{a^n b^l c^{n+l} : \; n, l \geq 0\}$$



Regular languages

Theorem: The language

$$L = \{a^n b^l c^{n+l} : \ n, l \geq 0\}$$

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Proof: Use the Pumping Lemma

$$L = \{a^n b^l c^{n+l} : \ n, l \geq 0\}$$

Assume for contradiction that L is a regular language

Since L is infinite we can apply the Pumping Lemma

$$L = \{a^n b^l c^{n+l} : \ n, l \geq 0\}$$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$ and

length $|w| \geq m$

We pick $w = a^m b^m c^{2m}$

Write $a^m b^m c^{2m} = x\ y\ z$

From the Pumping Lemma

it must be that length $|x\ y| \le m, \quad |y| \ge 1$

$$xyz = \overbrace{a\ldots aa\ldots aa\ldots a}^{m}\overbrace{b\ldots b}^{m}\overbrace{c\ldots cc\ldots c}^{2m}$$

$$xyz = \underbrace{a\ldots a}_{x}\underbrace{a\ldots a}_{y}\underbrace{a\ldots ab\ldots bc\ldots cc\ldots c}_{z}$$

Thus: $y = a^k, \quad k \ge 1$

$$x\ y\ z = a^m b^m c^{2m} \qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $x\ y^i\ z \ \in\ L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\ y^0\ z\ = xz\ \ \in L$

$$x\ y\ z = a^m b^m c^{2m} \qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $xz \in L$

$$xz = \underbrace{\overbrace{a...aa...a}^{m-k}}_{x\ \ z}\overbrace{b...b}^{m}\overbrace{c...cc...c}^{2m} \in L$$

$$xz = \underbrace{a...a}_{x}\underbrace{a...ab...bc...cc...c}_{z} \in L$$

Thus: $a^{m-k} b^m c^{2m} \in L$

$$a^{m-k}b^{m}c^{2m} \in L \qquad k \geq 1$$

BUT: $L = \{a^{n}b^{l}c^{n+l} : \; n,l \geq 0\}$

$$\Downarrow$$

$$a^{m-k}b^{m}c^{2m} \notin L$$

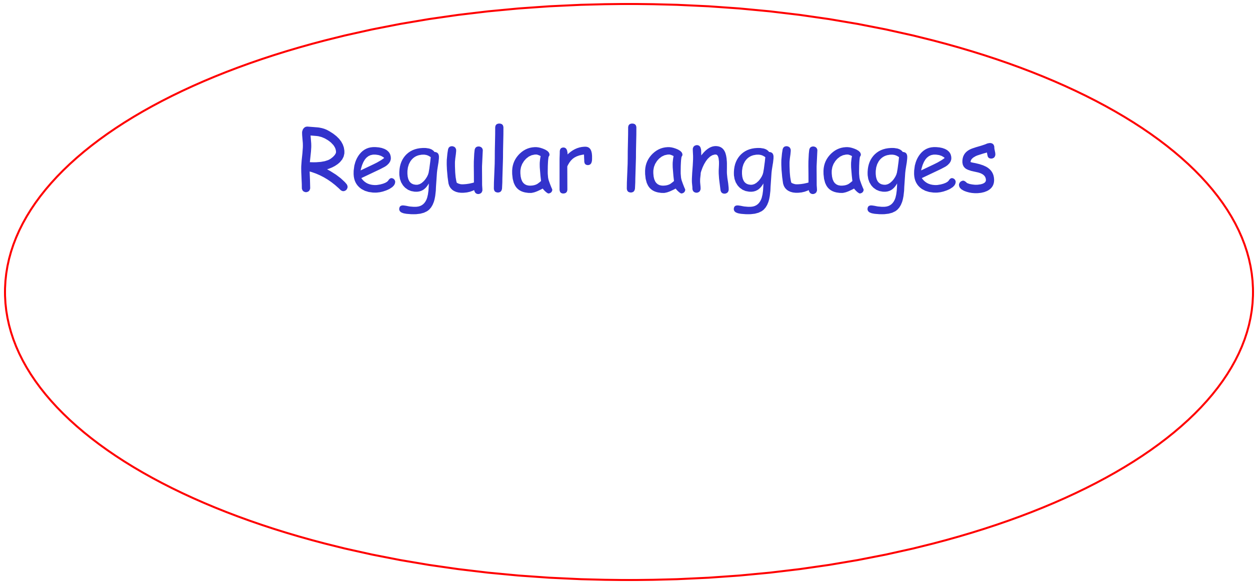
CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

Non-regular languages

$$L = \{a^{n!} :\ n \geq 0\}$$



Regular languages

Theorem: The language $L = \{a^{n!} : \ n \geq 0\}$

is not regular

$$n! \ = \ 1 \cdot 2 \cdots (n-1) \cdot n$$

Proof: Use the Pumping Lemma

$$L = \{a^{n!} : \ n \geq 0\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{a^{n!} :\ n \geq 0\}$$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$

length $|w| \geq m$

We pick $w = a^{m!}$

Write $a^{m!} = x\ y\ z$

From the Pumping Lemma

it must be that length $|x\ y| \leq m, \quad |y| \geq 1$

$$xyz = a^{m!} = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}a...a}^{m}\overbrace{a...aa...a}^{m!-m}$$

z spans from after y to the end:

$$\underbrace{a...aa...aa...a}_{z}$$

Thus: $y = a^k, \quad 1 \leq k \leq m$

$x\ y\ z = a^{m!}$ $\qquad$ $y = a^k, \quad 1 \le k \le m$

From the **Pumping Lemma:** $\quad x\ y^i\ z \quad \in\ L$

$$i = 0, 1, 2, \ldots$$

Thus: $\quad x\ y^2\ z \quad \in\ L$

$x\,y\,z = a^{m!}$ $\qquad$ $y = a^{k},\ \ 1 \le k \le m$

From the Pumping Lemma: $\quad x\,y^{2}\,z \ \in\ L$

$$xy^{2}z = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}\mathrlap{\underbrace{}_{z}}a...a}^{m+k}\overbrace{a...aa...a}^{m!-m} \ \in\ L$$

Thus: $\qquad a^{m!+k} \ \in L$

$$a^{m!+k} \in L \qquad 1 \le k \le m$$

Since: $L = \{a^{n!} : n \ge 0\}$

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There must exist p such that:

$$m! + k = p!$$

However: $m!+k \le m!+m$ for $m>1$

$$\begin{aligned} &\le m!+m! \\ &< m!m+m! \\ &= m!(m+1) \\ &= (m+1)! \end{aligned}$$

$$\Downarrow$$

$$m!+k<(m+1)!$$

$$\Downarrow$$

$m!+k \ne p!$ for any p

$$a^{m!+k} \in L \qquad 1 \leq k \leq m$$

BUT: $L = \{a^{n!} : \; n \geq 0\}$

$$\Downarrow$$

$$a^{m!+k} \notin L$$

CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

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